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Multi-parameter polynomial approximations for network synthesis

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

I. INTRODUCTION

In most electrical filter design, the major consideration is the frequency response characteristic. The design problem may be simplified to finding a physically realizable electrical network which has a specified response over a given frequency range. With the increased use of active elements, even the requirement on physical realizability is being relaxed to a large extent and now filters are being designed which at one time would have been considered unrealizable (1, 2, 3).

There is, however, another class of networks which are becoming more common and which do not share the relaxed physical realization criteria of the purely electrical filters. These are the nonelectrical networks which include mechanical or acoustic elements in either lumped or distributed form. These networks may represent a filter or a transducer, but they all contain frequency sensitive elements analogous to electrical inductance and capacitance. In general, these nonelectrical networks and some special electrical networks may include within their physical realizability conditions certain inter-component constraints (4).

These inter-component constraints may require special treatment when the device is to be designed on the basis of its lumped parameter electrical network. Since the synthesis techniques for lumped electrical networks are well developed, it is often desirable to express the synthesis of more general forms in terms of this analog. However, these synthesis techniques do not consider the inter-component constraints which may be present, so additional techniques may be required.

One purpose of the thesis is to develop approximation techniques which will allow the standard methods of electrical network synthesis to be applied to networks which contain inter-component constraints. This purpose may be expanded to include methods of approximation which will permit a wider variety of networks, all of which have similar frequency responses, to be synthesized. A second purpose for this type of approximation method is to permit a wider range of functions, irrational as well as rational, to be approximated in a form which makes them available as possible network response characteristics.

A. Practical Design Problem

The development of approximation methods which would allow electrical network synthesis techniques to be applied to the design of nonelectrical networks with inter-component constraints became necessary in the design of a miniature acoustic transducer.

Because of the acoustic nature of the elements and the small size of the transducer, several inter-component constraints had to be placed on the network elements. Also, since the device was being designed for a specialized application, rigid specifications were placed on the frequency response characteristics. The final configuration of the acoustic device specified the topology of the electrical analog network. Existing methods of approximation failed to produce networks which satisfied all of the above conditions. Therefore, to satisfactorily complete the transducer design, a new method of approximation had to be developed.

The design of the transducer discussed above and the particular constraints on this design are fully discussed in Section III of this thesis.

B. Proposed Solution

As a method of solving the problem of network synthesis with inter-component constraints, it was proposed to form a new method of approximating the desired response characteristic. This new method is to produce a family of approximating functions all having similar response characteristics, but differing in actual numerical values. In this way, a family of networks could be formed, all with similar responses, but with different element values. From this family those networks satisfying the constraints could be chosen.

Several fundamental decisions were made concerning the form of the approximation technique. The approximating function is to be an even polynomial and the Method of Gewertz (5) will be included in the synthesis procedure. Based on the work of Chebyshev in (6), it was decided to use an equi-ripple approximation to the function being approximated. In order to obtain a free parameter which could be varied independently of the pass band frequency response to give a family of networks, one of the pass band ripples was sacrificed. Thus, the original proposed solution was to approximate the needed response characteristic with an approximately Chebyshev polynomial with one pass band ripple sacrificed to allow an independently variable parameter. This proposal was later expanded to permit more than one ripple to be sacrificed allowing more than one variable to be used.

Two approximation methods were developed based on this proposal. The first of these, the extremum method, is discussed in Section II. This method is limited in scope allowing only a multi-parameter approximation of zero over the frequency range from zero to one. This method proved successful in the acoustic transducer design problem. The second method, discussed in Sections IV, V, and VI, is the coefficient method which is a general method for approximating any single-valued, finite function over a given range. This more general method was developed from ideas used in the work with the extremum method.

C. Comparison with Existing Methods

There are several approximation techniques which are now being used in network design problems. The best known of these are the methods of Chebyshev and Butterworth (3). Both of these methods approximate the function to within limits by a polynomial. However, once definite specifications are made, only one polynomial is available and no variation within the method is possible.

Two other methods of interest are more recent than the first two discussed. The approximation method developed by N. B. Jones (7) makes use of the Chebyshev Polynomials and by frequency scaling techniques permits one ripple location to be specified. In this way a number of polynomials may be formed, all having approximately the same response as the original Chebyshev Polynomial. Using this method only small variations about the original may be achieved. The only parameter available is the ripple location. This method is also limited to approximations of a constant.

The approximation method of major interest for comparison with the methods of this thesis is that of D. S. Humpherys (8). This method allows any polynomial to be approximated in an equi-ripple manner by a rational function. There exists the possibility of using one root location of the rational function as a variable parameter by sacrificing one pass band ripple. The method does allow a closed form solution once a given set of conditions is met and requires only the solution of a set of linear equations. The major limitation on this method is the requirement that the function to be approximated be expressible as a polynomial.

Each of the existing methods fails to meet all the requirements desired in the methods of this thesis. These methods are to permit an approximation of any function or curve to within limits of magnitude and over a range of frequency. Furthermore, the approximation is to be accomplished by a family of polynomials produced by variation of one or more arbitrary parameters. The parameters are to be extra conditions which may be imposed as a result of the sacrifice of an appropriate number of pass band ripples.

A recent article by Ishizaki and Watanabe (9) describes a technique for optimizing a network in order to approximate a Chebyshev type response. Examination of the system of equations used in this technique leads to the conclusion that the approximation methods described in this thesis may be adaptable for use in network optimization as well as for the function approximation purpose originally outlined.

II. THE EXTREMUM METHOD

This section and the one following describe the extremum method of approximating a constant and illustrate its application to the design of an acoustic transducer. The concepts underlying the extremum method and its development are discussed in this section. Section III includes details of the acoustic transducer design problem and its solution using the extremum method.

In order to simplify the notation to be used, the term "extremum" will be used to signify that value of x at which the polynomial, $P(x)$ has a local minimum or maximum. The polynomial, $P(x) = x^2 + 1$, is a minimum at $x = 0$ and thus zero is an extremum of this polynomial. The value of $P(x)$ at the extremum will be referred to as the value at the extremum. In the example polynomial the value at the extremum is one.

A. Fundamental Concepts

The fundamental concepts underlying the extremum method are similar to those upon which the Chebyshev Polynomials are based. These polynomials are commonly used to approximate a constant in an equi-ripple manner over a finite range of the independent variable. Figure 1 shows a typical zero pattern of a Chebyshev Polynomial and its behavior along the real axis of the x-plane. These polyno-

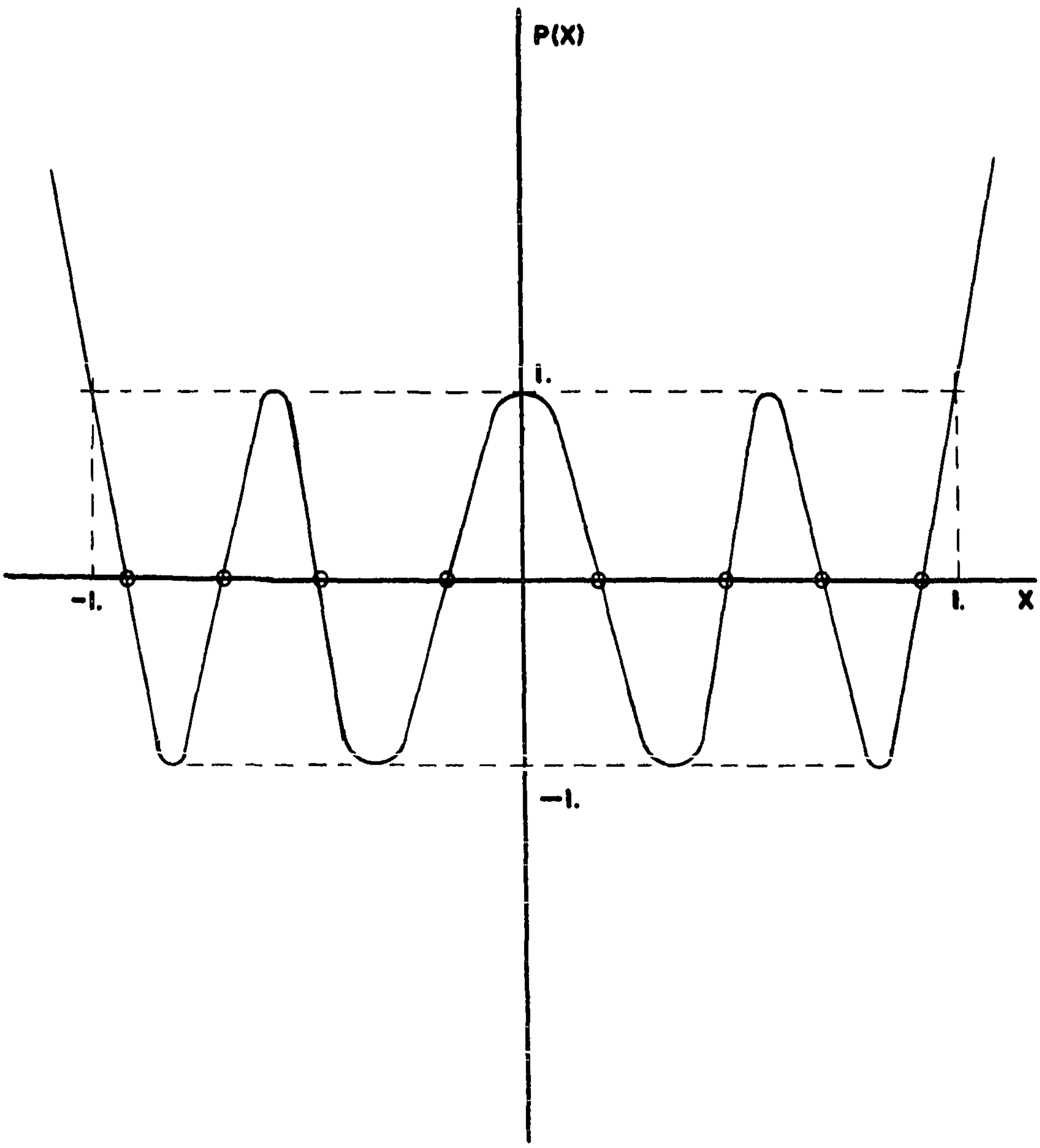


FIGURE 1. REAL AXIS BEHAVIOR AND ZEROS OF EIGHTH ORDER CHEBYSHEV POLYNOMIAL

mials have either even or odd symmetry about the vertical axis.

In order to simplify the equations and the computer programming a change in the independent variable has been made. As used in network synthesis the polynomials would be a function of complex frequency, s, and their zeros would lie on the imaginary axis of the s-plane. Throughout this thesis the new variable, $x = -js$, is used to force the zeros of $P(x)$ to lie on the real axis.

1. Basic Chebyshev specifications

The example Chebyshev Polynomial, characteristics of which are plotted in Figure 1, is an eighth order polynomial of the form given in Equation 1. In this form, there are five unknowns, the five a_i coefficients, which must be determined by specifying five conditions on the polynomial.

$$a_8x^8 + a_6x^6 + a_4x^4 + a_2x^2 + a_0 = P(x) \qquad (1)$$

In the case of the Chebyshev Polynomials, the five conditions are the two end conditions, $P(0) = 1$ and $P(1) = 1$, and three extrema. The conditions at the extrema are determined by the equi-ripple requirement and may be specified by requiring that the values at the extrema be ± 1. The end point condition at zero may be considered a special case of the extrema conditions for even polynomials.

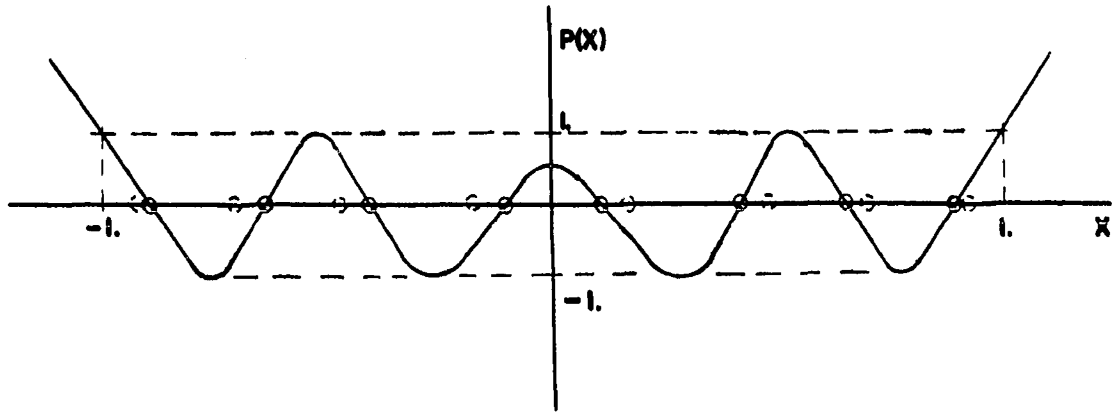
These end point and extrema conditions, when applied to a general even polynomial of the form given in Equation 2 result in an equi-ripple approximation to zero over the

$$P(x) = \sum_{i=0}^{n} a_{2i}x^{2i} \tag{2}$$

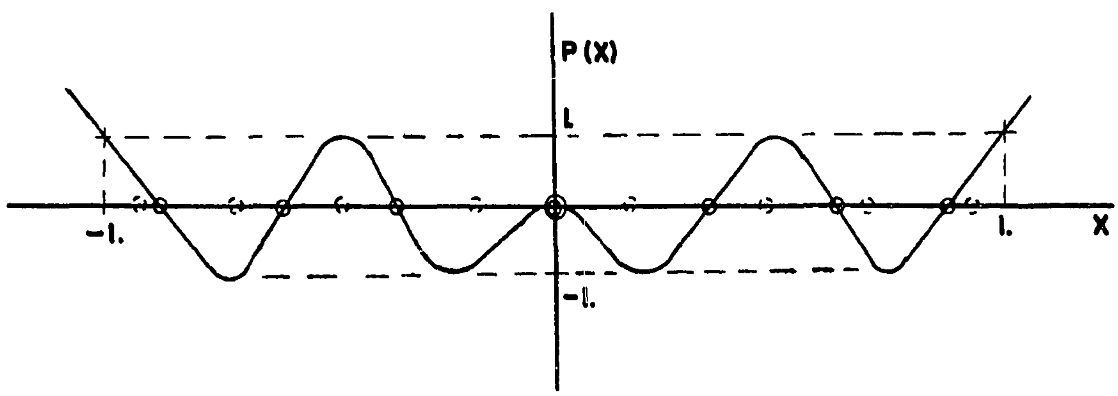
range from $x = 0$ to $x = 1$. This approximating polynomial will have n zeros lying on the real x-axis between zero and one and n extrema including zero in this same range. This general even polynomial will have $n + 1$ unknown coefficients. These unknown coefficients can be determined by specifying conditions on the n extrema and the end point at $x = 1$. These conditions determine the Chebyshev Polynomial of order 2n.

2. Effect of relaxing condition at zero

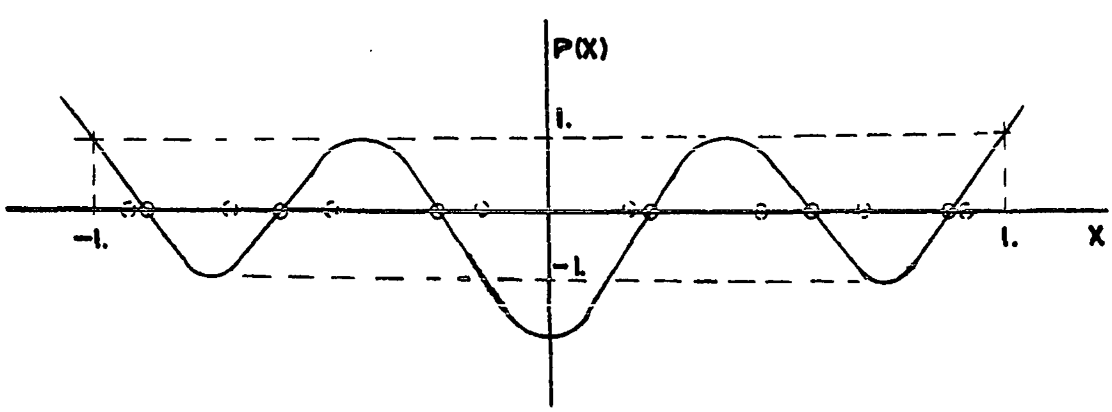
A set of modified polynomials may be formed from the Chebyshev Polynomials by allowing the condition at zero to be other than $P(0) = 1$. Figure 2 illustrates the effects of allowing $P(0)$ to be less than one but greater than zero, to be zero, and to be less than zero. In each case the polynomial zeros are shifted from the zeros of the original Chebyshev Polynomial represented by the dotted circles. As $P(0)$ decreases toward zero, the real axis zeros are forced onto the imaginary axis. As $P(0)$ becomes more negative a pair of extrema also become imaginary and both the real axis extrema and zeros between $x = 0$ and $x = 1$ are reduced in number by one.



A. P(O) LESS THAN 1. BUT GREATER THAN ZERO



B. P(O)=O, DOUBLE ZERO AT ORIGIN



C. P(O) LESS THAN -1. TWO ZEROS ON IMAGINARY AXIS

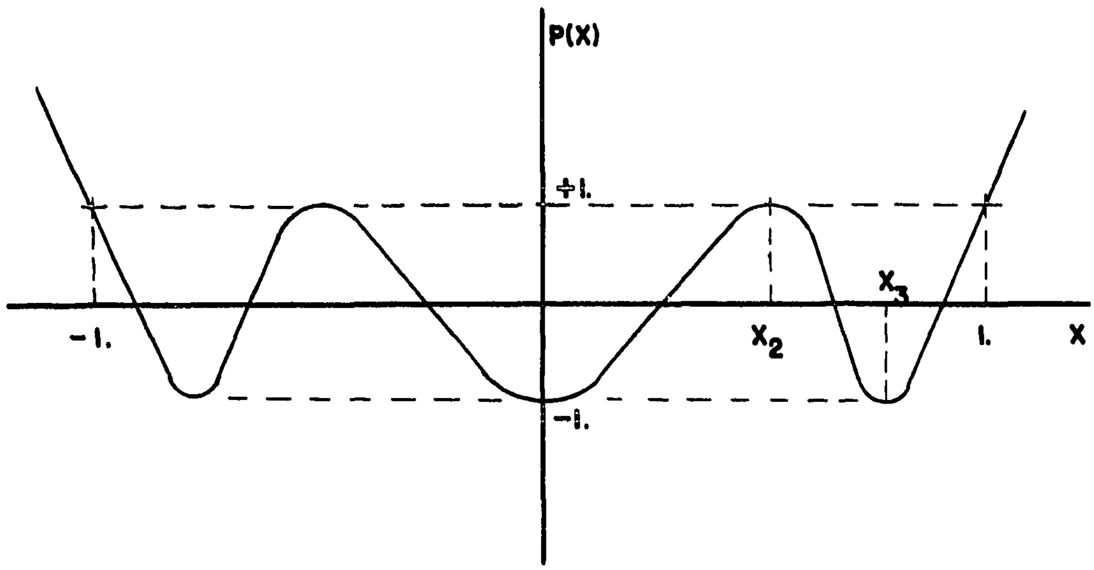
FIGURE 2. REAL AXIS BEHAVIOR AND ZEROS OF MODIFIED EIGHTH ORDER POLYNOMIALS

ORIGINAL ZEROS SHOWN AS DOTTED CIRCLES

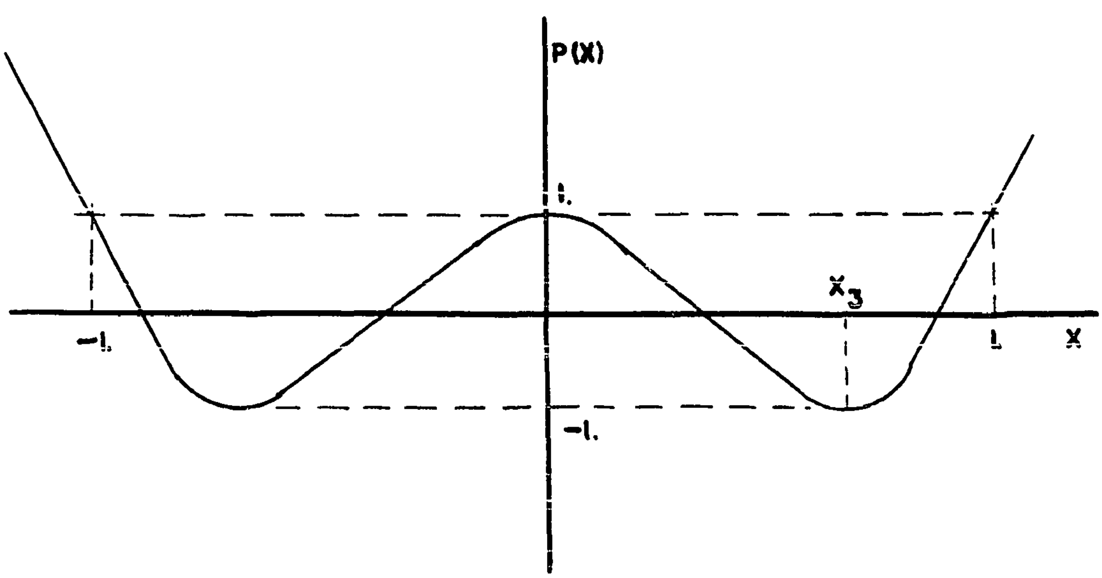
3. Imaginary axis extrema

A logical extension of the results of decreasing the value of P(0) is to allow this value to become equal to -1. In this case, as shown in Figure 3a, the behavior of this modified polynomial resembles that of the Chebyshev Polynomial of order 2(n - 1) even though the actual polynomial is of order 2n. The polynomial is still an equi-ripple approximation to zero but with one fewer pass band ripples than the original Chebyshev approximation. Also, the slope of the curve as it passes through the end point has been reduced due to the shift toward the origin of the zero closest to one, the slope of the curve as it passes through the end point has been reduced.

Because the real axis behavior of this modified equi-ripple approximating polynomial is similar to that of the next even order Chebyshev Polynomial of lower order, one fewer condition must be specified to preserve this character. Since n + 1 conditions are still needed to specify the n + 1 coefficients of the polynomials and only n are needed to specify the equi-ripple approximation to zero specification, there is one condition which may be specified arbitrarily. This extra condition allows the introduction of an arbitrary parameter which may be selected independently of the desired frequency response requirements and permits a family of networks, all having



A. ONE PAIR OF COMPLEX ZEROS, EXTREMUM X_1 ARBITRARY



B. TWO PAIRS OF COMPLEX ZEROS, EXTREMA X_1, X_2 ARBITRARY

FIGURE 3. REAL AXIS BEHAVIOR OF MODIFIED EIGHTH ORDER POLYNOMIALS WITH EQUI-RIPPLE CHARACTERISTICS

the same frequency response character, to be formed.

This analysis may be extended to include more than one arbitrary parameter if more than one pass band ripple is sacrificed. This possibility is illustrated in Figure 3b in which the pass band behavior of the polynomial having two pass band ripples sacrificed is shown. As in Figure 3a, this represents an eighth order polynomial designed to be an equi-ripple approximation to zero. Since there are now four zeros located symmetrically with respect to the origin within the x-plane, no general statement may be made concerning the direction of shift of the remaining real axis zeros and the change of slope at $x = 1$.

The extremum method is based on the analysis above. The conditions imposed upon the polynomials to be developed by this method are essentially those described for the Chebyshev Polynomials with the exception that one or more extrema are forced off the real axis. These complex extrema are used as the arbitrary parameters in the formation of a family of polynomials.

B. Initial Hypotheses

The development of the extremum method was based on the fact that the polynomials developed by this method must be even order and be equi-ripple approximations to a constant. One or more extrema pairs of these polynomials must be complex and their location within the x-plane may be arbitrary.

1. Even polynomials

The polynomials being considered throughout this thesis are even polynomials of the form given in Equation 2. In this equation, x is related to the complex frequency, s, commonly used in network synthesis by the relationship, $x = -js$. The polynomial was chosen to be even since the synthesis procedure uses the Method of Gewertz as mentioned in the Introduction.

2. Extrema locations

An additional hypothesis concerns the location of the extrema of the polynomials. In order to ensure that there will be no significant frequency components passed by the network outside the specific pass band, there must be no real axis extrema of $P(x)$ outside the specified range of approximation. Extrema may exist off the real axis anywhere within the complex x-plane.

3. Network conditions

Several conditions are, of course, specified by the network itself, both in terms of its physical realizability and of its inter-component constraints. The realizability constraint is, for most practical purposes, relieved by the synthesis procedure to be used and thus is not included as a factor in the approximation technique.

One purpose behind the desire for a family of polynomials is the ability to choose one particular polynomial from the family on the basis of satisfying the constraints. In this way the constraints may be satisfied without specifically including them in the approximation procedure. For this reason, none of the many possible constraints are considered in the development of the approximation techniques although, as will be pointed out in a later section, some constraints can be considered when the arbitrary parameters are defined.

C. Description of the Extremum Method

The purpose of the extremum method of approximation is to develop a multi-parameter family of polynomials which approximate a constant over a given range of x. These polynomials are essentially variations of the usual Chebyshev polynomials with the x-plane root locations changed as shown in Figure 3. The notation of Figure 3 will be used throughout this thesis.

1. Extrema

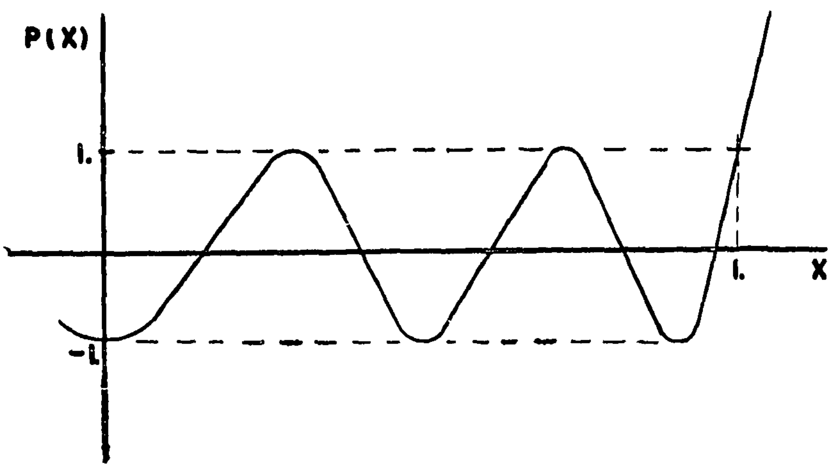
Since the extremum method is primarily designed for equi-ripple approximations, the maxima of P(x) will have the value +1 and the minima the value (- 1). For an m-parameter family x_1 through x_m are the arbitrary extrema and x_{m+1} through x_{n-1} the extrema along the real axis. These extrema define Equation 3.

$$
\begin{array}{llll}
P(x_1) = K_1 & & P'(x_1) = 0 \\
---- & & ---- \\
P(x_m) = K_m & & P'(x_m) = 0 \\
P(x_{m+1}) = \pm 1 & & P'(x_{m+1}) = 0 \\
---- & & ---- \\
P(x_{n-1}) = \pm 1 & & P'(x_{n-1}) = 0
\end{array}
$$

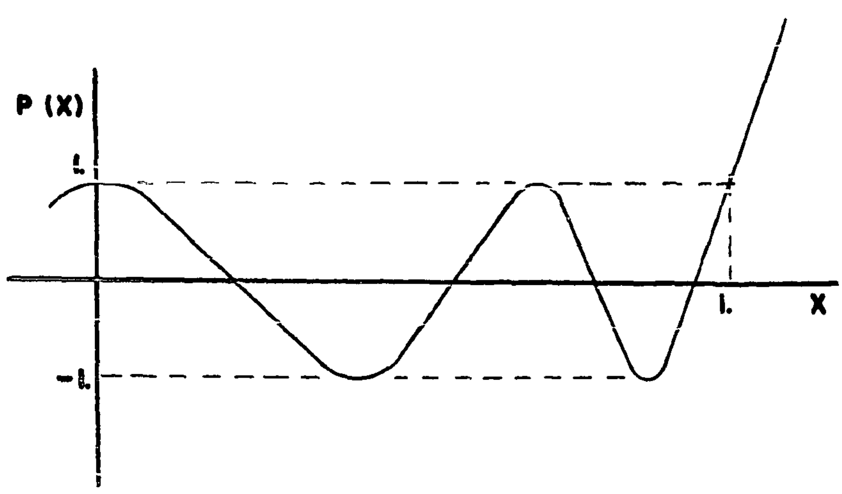
2. <u>End conditions</u>

In addition to the extrema conditions described above, two end conditions are required before all unknown coefficients are completely specified. These end conditions are $P(0) = \pm 1$ and $P(1) = \pm 1$. Usually the sign of the end condition for $x = 1$ is specified and the sign of $P(0)$ is determined by the number of ripples in the pass band. Figure 4 illustrates the interrelationship between the number of pass band ripples and the end point signs.

Since the number of pass band ripples for a Chebyshev



A. EVEN NUMBER OF RIPPLES, P(0) = -1, P(1) = 1



B. ODD NUMBER OF RIPPLES, P(0) = 1, P(1) = 1

FIGURE 4. ILLUSTRATION OF RELATIONSHIP BETWEEN END POINT CONDITIONS AND NUMBER OF RIPPLES

Polynomial of order 2n is n and the number of arbitrary parameters is m, the end point signs are related in a manner depending upon whether n-m is even or odd. In Figure 4a, the number of pass band ripples, n-m, is even and the end points have opposite signs. If the number of pass band ripples is odd, as in Figure 4b, the end points have the same sign. Therefore, if one end condition is specified, the other is determined by Equation 4.

$$P(0) = (-1)^{n-m-1}P(1) \tag{4}$$

3. Normalization

The preceding discussion has dealt with polynomials which are approximations to zero with a unit ripple magnitude over the range of x from zero to one. Because of scaling techniques, no generality is lost by this normalization. The range of x over which the approximation is valid may be extended by frequency scaling.

For network synthesis, the polynomials will normally be used in the form given in Equation 5. In this form the

$$D(x) = 1 + rP(x) \tag{5}$$

ripple magnitude may be controlled by varying the ripple factor, r, and the constant being approximated can be controlled by magnitude scaling.

D. Development of the Extremum Method

Several methods were used in attempts to develop an explicit form for the approximation technique. Two unsuccessful attempts are outlined in Appendices A and B. Both of these methods were abandoned when it was found that computational difficulties forced a series of assumptions. These assumptions restricted the form of the final solution to such an extent that no practical polynomial families could be formed. It may be possible that additional investigation along the lines of the methods outlined in the Appendices might result in some workable solutions.

The method which proved practical in terms of producing usable polynomials is a numerical method involving the solution of a system of nonlinear equations. The basic steps of the method include the steps outlined here and discussed in detail in the following sections.

1. Formation of extrema estimates
2. Formation of derivative polynomial, $P'(x)$
3. Integration of derivative polynomial $P'(x)$ to obtain $P(x)$
4. Evaluation of a constant of integration, L
5. Evaluation of multiplicative constant, k
6. Evaluation of $P(x)$ at estimated extrema
7. Determination of extrema error
8. Improvement of extrema estimates

1. Extrema estimates

The input to each cycle of computations is a set of extrema estimates. The input matrix X is defined by Equation 6. This is the form for an m-parameter family of polynomials. The first m extrema pairs are used as the arbitrary parameters and are constants through the computational cycles

$$[X] = \begin{bmatrix} x_{m+1} \\ x_{m+2} \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \end{bmatrix} \qquad (6)$$

for the particular polynomial being formed.

Initially, in the iterative process, these estimates must be supplied externally. Once into the iterative process, the estimates are supplied for each cycle by the previous cycle until the estimates are refined sufficiently to be considered a solution.

2. Formation of the derivative

The derivative polynomial, $P'(x)$, is formed using Equation 7. In this equation all m arbitrary parameter pairs have been defined along the imaginary axis. If an

$$P'(x) = kx(x^2 + x_1^2)(x^2 + x_2^2) \ldots (x^2 + x_m^2)(x^2 + x_{m+1}^2) \ldots (x^2 - x_{m-1}^2) \qquad (7)$$

off-axis extrema combination is preferred, the term $(x^4 + ax^2 + b)$ is substituted for two of the $(x^2 + x_i^2)$ terms. The constants a and b define the complex extrema. The factor x is a result of the even x nature of the final polynomial. k is a constant to be defined later.

Equation 7 is then multiplied out to obtain the polynomial form of Equation 8. In this equation and the subsequent ones, the coefficients, C_i, are nonlinear functions of the arbitrary real extrema, x_1 through x_m, and the

$$P'(x) = k(C_1 x^{2n-1} + C_3 x^{2n-3} + \dots + C_n\, x) \qquad (8)$$

extrema estimates, x_{m+1} through x_{n-1}.

3. <u>Integration</u>

To obtain P(x), Equation 8 is integrated. The form of P(x) which is the result of this operation is shown in Equation 9. In this equation, k is a multiplicative

$$P(x) = k\left[\frac{C_1}{2_{n-1}}\, x^{2n} + \frac{C_2}{2_{n-3}}\, x^{2n-2} + \dots + \frac{C_n}{2}\, x^2\right] + L \qquad (9)$$

constant which is used to satisfy the requirements for P(1). L is a constant of integration which is used to satisfy the requirements for P(0).

4. Constant of integration

In this case the constant of integration, L, can be evaluated by inspection of Equation 9 and found to be equal to P(0). For the general case, Equation 4 can be used to calculate this value for any problem.

5. Multiplicative constant

It is necessary to include a multiplicative constant, k, in the expression for P(x). Since k is not dependent upon either the extrema estimates or the arbitrary extrema, it may be chosen to satisfy some other polynomial property.

For purposes of this development, the value of k is chosen to satisfy the end condition given in Equation 4. For approximations with an even number of pass band ripples (i.e., P(0) = -P(1)) the value of k can be found using Equation 10. The sign is chosen to be the same as that

$$k = \pm 2/\left(\frac{C_1}{2_{n-1}} + \frac{C_2}{2_{n-3}} + \dots + \frac{C_n}{2}\right) \tag{10}$$

of P(1). For approximations with an odd number of pass band ripples (i.e., P(0) = P(1)), any value of k will satisfy the end condition. Therefore the value of k determined by using Equation 10 can be used for both cases.

6. Polynomial evaluation

The polynomial, in its final form given in Equation 9, is evaluated at the estimated extrema producing a test matrix Y defined in Equation 11. This vector is used to

$$[Y] = \begin{bmatrix} P(x_{m+1}) \\ P(x_{m+2}) \\ \cdot \\ \cdot \\ \cdot \\ P(x_{n-1}) \end{bmatrix} \tag{11}$$

determine whether the polynomial produced is a solution or whether another computational cycle must be begun.

7. Test for error

From the discussion in Section II, C, 1, it is apparent that once the approximating polynomial has been formed, Y should consist only of ones with alternating signs. This ideal matrix, Y_1, is compared to the actual Y matrix to obtain a measure of the error in the iterative process producing the approximating polynomial. One possible measure of the error is given in Equation 12. This particular method of defining the error is a measure of the variations of the extrema magnitudes from the limits.

$$\text{Error} = \sum_{i=m+1}^{n-1} [P(x_i) - \text{sgn}\,(P(x_i))]^2 \tag{12}$$

Once the error measure is less than some specified maximum, the polynomial is defined as a solution.

8. Estimate improvement

If the error measure, as computed by Equation 12, is above the maximum allowable error, the estimates of the extrema locations must be improved and the cycle repeated. The output matrix Y, as well as the original input, X, are used in the improvement process.

An improved set of estimates is obtained by solving Equation 13 for the values of x which do satisfy the extrema conditions.

$$[Y] - [Y_1] = 0 \tag{13}$$

Since the coefficients of $P(x_{m+k})$ are nonlinear functions of x_1 through x_{n-1}, Equation 13 is a set of (n-m-1) nonlinear equations in the unknown extrema.

There are several methods of solving systems of nonlinear equations which may be used to produce an improved set of estimates (6, 10). Among the better known are the methods of Newton-Raphson, Wegstein, and Muller (10). In practice it was found that this part of the computation cycle gave the most trouble since most commonly used methods of solving nonlinear systems require accurate initial estimates for convergence.

E. Practical Considerations

As mentioned above, one of the major difficulties is the tendency of the methods of improving the extrema estimates to diverge unless an accurate set of initial estimates is supplied. In many cases, such initial estimates may be obtained using Chebyshev polynomials.

The initial estimates for a polynomial of order 2n having m arbitrary parameters may be obtained by using as estimates the extrema of the Chebyshev Polynomial of order (2n-2m). If the values of the m arbitrary parameters are sufficiently large, the extrema of the lower order Chebyshev polynomial are close enough to the approximation polynomial extrema to ensure convergence to a solution. Once the extrema of a first polynomial are known, they may be used as the initial extrema estimates for a new polynomial having slightly different arbitrary parameters.

An example of the results of this procedure is illustrated by the curves of Figure 5. The polynomials being developed were 12th order polynomials with one arbitrary parameter. This figure shows a family of curves representing the changes in the four real extrema as the magnitude of the imaginary axis extrema is varied. The initial magnitude of the parameter x_1 was set to ten and the extrema of the tenth order Chebyshev polynomial were used as the initial estimates. The procedure converged to

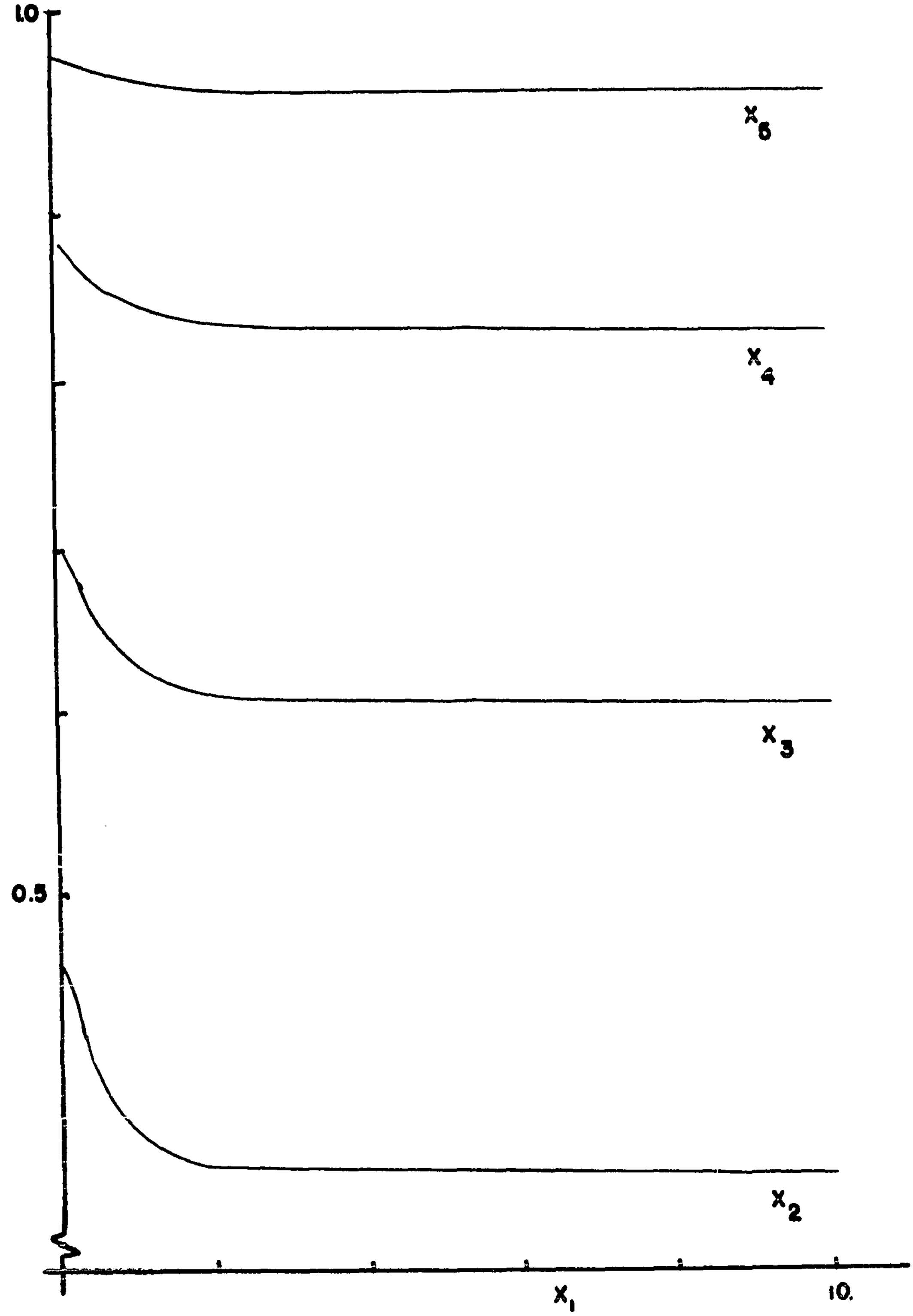


FIGURE 5. VARIATION OF NON-ZERO EXTREMA FOR TWELFTH ORDER POLYNOMIALS WITH ONE ARBITRARY PARAMETER

a solution. The parameter was then reduced to five and the extrema of the previous polynomial were used as the initial estimates. Again, the procedure converged to the solution. This procedure was repeated until x_1 reached zero. As can be seen from Figure 5, very small changes in extrema are occurring for $x_1 \geq 1$ permitting rather large steps in x_1 to be made. Once x_1 becomes less than one, large changes in the extrema locations occur and the step size must be reduced to permit convergence to a solution within a reasonable number of iterations.

Using this method of determining extrema estimates, a family of polynomials can be produced faster than one particular polynomial can be if no accurate estimates are available. In addition, since the curves of extrema variation with arbitrary parameter are smoothly varying, polynomial regression techniques may be used to form a polynomial representation of each extrema as functions of the arbitrary parameter. Using these polynomials, the extrema for any value of the arbitrary parameter may be calculated.

F. Comparison with Other Methods

Before the extremum method was developed, several other methods were used in attempts to synthesize networks with inter-component constraints. The first attempts used Chebyshev and Butterworth polynomials. These methods produced physically realizable electric networks of the proper form, but which did not meet the component constraints imposed. Elliptic functions (1, 2, 6) were also tried, but required changes in the topology of the network. None of the methods produced a family of polynomials formed through use of an arbitrary parameter.

The method of approximating a constant developed by N. B. Jones (7) as outlined in Section 1.C is similar to the extrema method in some respects. His method consists of translating the Chebyshev polynomials along the axis, thus permitting the frequency at which one extremum may occur to be specified. By this translation it is possible that one or more pass band ripples may be shifted into the negative frequency region. Jones' method, though simpler, is not as versatile and does not permit as wide a range and choice of parameters as the extremum method.

G. General Remarks

Use of the extremum method as described in this section permits the development of a multiple-parameter family of polynomials which are equi-ripple approximations to a constant over the range from zero to one. Because it uses the extrema of the polynomials as the unknowns, the method requires the solution of only (n-m-1) nonlinear equations and (m+2) linear equations in the development of an even polynomial of order 2n. The linear equations represent the two end points and the m arbitrary parameters. The nonlinear equations represent the pass band extrema.

The extrema were chosen as the unknowns in the system in an attempt to reduce the number of nonlinear equations and thereby reduce the difficulties in computation. However, this choice of unknowns produced equations which were highly nonlinear. This high degree of nonlinearity increased the difficulties in the computation, especially in terms of the tendency of the method to diverge.

As is illustrated in Section III, the extremum method does work well in practice and can produce practical network designs in cases where the traditional methods failed to do so.

III. APPLICATION TO TRANSDUCER DESIGN

The extremum method of approximating a constant by a multi-parameter family of polynomials was developed in reponse to an actual industrial problem, the design of an electro-acoustic transducer. This section outlines that practical application of this type of approximation technique.

Because of the nature of the miniature acoustic device being designed, specifications other than frequency response had to be satisfied. For this reason, standard approximation methods which are based on frequency response criteria only were not appropriate and the extremum method, producing a family of polynomials with similar frequency response characteristics, was developed. This family of polynomials yields a family of networks with the desired frequency response, from which the member best satisfying all of the specifications can be chosen.

The additional requirements, other than the frequency response requirements, were dictated by the small size and the nonelectrical nature of the final device. Several important inter-component constraints were specified by the types of acoustic elements which comprise the final device. Because of the small size of the transducer, approximately 1/8 inch by 1/8 inch by 1/4 inch, other constraints were imposed upon the final acoustic network, primarily to make the device practically producible by mass production techniques. The basic acoustic configuration specified the topology of the electrical analog used in the synthesis process.

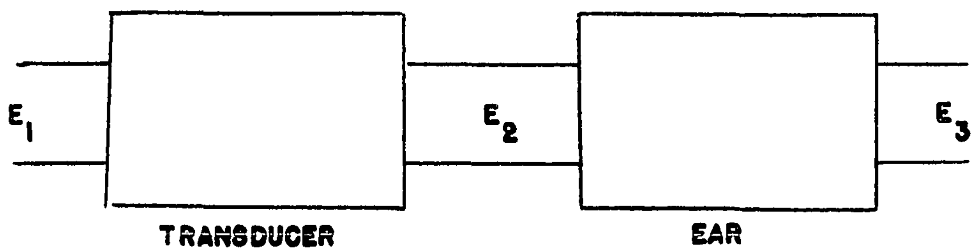
A. Transducer Requirements

The requirements on the transducer design may be divided into three categories. These are frequency response, analog electrical network topology, and inter-component constraints.

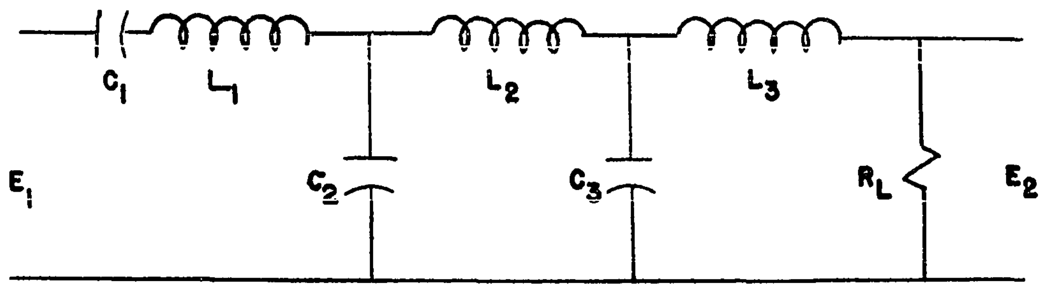
1. Frequency response

The fundamental requirement on the frequency response of the transducer was that, when operating into the human ear, the overall response be an equi-ripple approximation to a constant. The peak to valley magnitude of this ripple was to be three decibels.

The overall system consists of two parts as shown in the block diagram of Figure 6a, the transducer itself and the human ear. In this block diagram, voltages are used as the analogs of the various input and output signals. The overall transfer function, E_3/E_1, has been specified to be an equi-ripple approximation to a constant. Since that part of the human ear affecting the frequency response can be approximated as $E_3/E_2 = s$, the magnitude of the required transducer transfer function, E_2/E_1, must be an equi-ripple approximation to $1/s$.



A. BLOCK DIAGRAM REPRESENTATION



B. ELECTRICAL ANALOG OF TRANSDUCER

FIGURE 6. ANALOG REPRESENTATION OF ACOUSTIC SYSTEM

2. Network topology

Since the final device is an acoustic network, the topology of the electrical analog network is fixed by the physical configuration of the device. The electrical analog which was developed from the acoustic configuration is shown in Figure 6b. This network was specified independently of the frequency response and was based solely on the physical configuration of the device.

The load resistor indicated in the diagram includes losses within the acoustic components and the loading effects of the ear. The differentiating effect of the ear is not included within this network. Thus, the network represents only the transducer and its resistive load.

3. Inter-component constraints

Several constraints on the component values relative to those of other components were imposed on the electrical analog by the nature of the acoustic elements and the fabrication process. These inter-component constraints are:

1. When scaled to make the load resistor one ohm, the capacitor, C_2, must be larger in numerical value than the inductor, L_1.
2. When scaled to make the load resistor one ohm, the capacitor, C_3, must be larger in numerical value than the inductor, L_2.

3. The value of the input capacitor, C_1, must be equal to or larger than that for C_3.
4. The sum of the capacitances should be as small as possible. Since capacitance is a measure of volume, this ensures a small transducer volume.

The first two of these constraints are absolute requirements due to the nature of miniature acoustic elements (4). The latter two constraints are not absolute requirements, but are desirable in terms of ease of fabrication and miniaturization.

B. Design Procedure

The design consisted of developing a network which met all the requirements listed above. The first step in this design involved developing a family of polynomials which approximate the appropriate function giving the desired flat response. The second part of the procedure involved the use of the Method of Gewertz to produce network functions which could be reduced to electrical networks by standard synthesis techniques.

Since the Method of Gewertz was to be used, the function to be formed was the real part of the squared input admittance. The denominator of this function was of the form $1+rP(x)$, where the $P(x)$ polynomial was a twelfeth order approximation to zero as discussed in Section II. To obtain the desired integrating effect, numerator of the function was a frequency squared term. The ripple factor, r, had the value of 0.33 to produce the desired ripple magnitude. The formation of P(x) is not affected by the choice of the ripple factor.

The extremum method was used to form a one parameter family of polynomials which approximate zero over the range of x from zero to one. These polynomials and the desired ripple factor were used as inputs to a computer program which created the appropriate numerator, applied the Method of Gewertz and synthesized the electrical network. The resulting network family was analyzed and examined to determine those networks which satisfied all specifications.

C. Design Results

1. Results using other methods

Before the extremum method was developed for this problem, attempts were made to design the transducer using existing methods. The methods tried were those using Chebyshev and Butterworth Polynomials and later the more recent methods developed by N. B. Jones (7) and Deverl Humpherys (8). These methods failed to produce networks which met all specifications.

The network component values obtained using the twelfth order Chebyshev Polynomial are listed in Table 1. From this table it can be seen that, although the network is physically realizable as an electrical network, it is

Table 1. Chebyshev network components

$C_1 = 1.086$ f	$C_2 = 1.344$ f	$C_3 = 1.259$ f
$L_1 = 2.634$ h	$L_2 = 2.661$ h	$L_3 = 1.756$ h

not realizable as an acoustic network with the inter-component constraints described previously. Although this network is not a realizable transducer, it will be used as a basis for comparison for the networks developed using the extremum method.

2. <u>Results using the extremum method</u>

The design procedures outlined previously were applied to polynomials formed using the extremum method. The approximating polynomials were of the form given in Equation 14. The coefficients for the one parameter family are given in Table 2.

$$a_1x^{12} + a_2x^{10} + a_3x^8 + a_4x^6 + a_5x^4 + a_6x^2 + 1 = P(x) \tag{14}$$

This polynomial family was formed using the extremum of the polynomial located on the imaginary axis of the x-plane (the real axis of the s-plane) as the arbitrary parameter. As described in Section 11.D, a large value for x_1, the arbitrary parameter, was used to form the

Table 2. Polynomial family from extremum method

x_1	a_1	a_2	a_3	a_4	a_5	a_6
5.0	16.73305	461.7498	-1224.451	1092.712	-394.4147	49.67062
4.5	20.56558	450.2625	-1211.770	1086.486	-393.1399	49.59531
4.0	25.86320	434.3053	-1194.074	1077.764	-391.3480	49.48923
3.5	33.47606	411.4271	-1168.750	1065.296	-388.7864	49.33725
3.0	44.95219	377.0098	-1130.721	1049.595	-384.9456	49.10919
2.5	63.29808	321.6480	-1069.163	1016.137	-378.6533	48.73322
2.0	95.10191	225.5953	- 962.2121	963.0937	-367.6500	48.07110
1.5	156.4341	39.76873	- 745.4078	859.4108	-345.9530	46.74738
1.0	292.5809	- 376.1719	- 284.3729	621.6228	-295.2146	43.55585
0.9	338.1299	- 516.3444	- 124.4807	539.7075	-277.4150	42.40274
0.8	393.6276	- 687.7921	72.07461	438.3096	-255.1554	40.93575
0.7	461.6256	- 898.8974	315.6502	311.5542	-226.9699	39.03735
0.6	545.3663	-1160.410	619.7198	151.6269	-190.8386	36.53588
0.5	648.8170	-1485.778	1001.601	- 51.86849	-143.9449	33.17370
0.4	776.1910	-1890.157	1482.029	- 312.1969	- 82.42875	28.56307
0.3	931.1291	-2387.200	2080.825	- 643.1107	- 1.826661	22.18256
0.2	1109.258	-2965.550	2788.842	-1043.324	99.09057	13.68778
0.1	1276.406	-3514.960	3472.401	-1438.516	202.1804	4.489230
0.0	1350.934	-3761.862	3782.791	-1620.567	250.7054	0.000000

initial polynomial and the extrema of this polynomial were used as estimates for the succeeding polynomial. In Table 2 it requires approximately the same number of iterations to go from any polynomial to the one immediately following when the extrema of the former are used as the initial estimates for the latter.

The family of electrical networks corresponding to the polynomial family of Table 2 was synthesized using the ripple factor corresponding to a ripple factor of 0.33. The component values for this network family are given in Table 3. Since all members of this network family meet all transducer requirements, any member may be used in the final design. Therefore the final selection may be based on considerations other than those listed in this discussion. Typical of the characteristics which were considered in the final selection are cut-off characteristics, sensitivity to input signals and stability. In the event that this one parameter family had failed to produce satisfactory networks, it had been planned to sacrifice a second pass band ripple and form a two parameter family of networks.

The networks were examined for their frequency responses using computer generated frequency response plots. Figure 7 shows typical frequency response curves obtained for the complete system including these networks and the ear effects. The response curve for the Chebyshev network

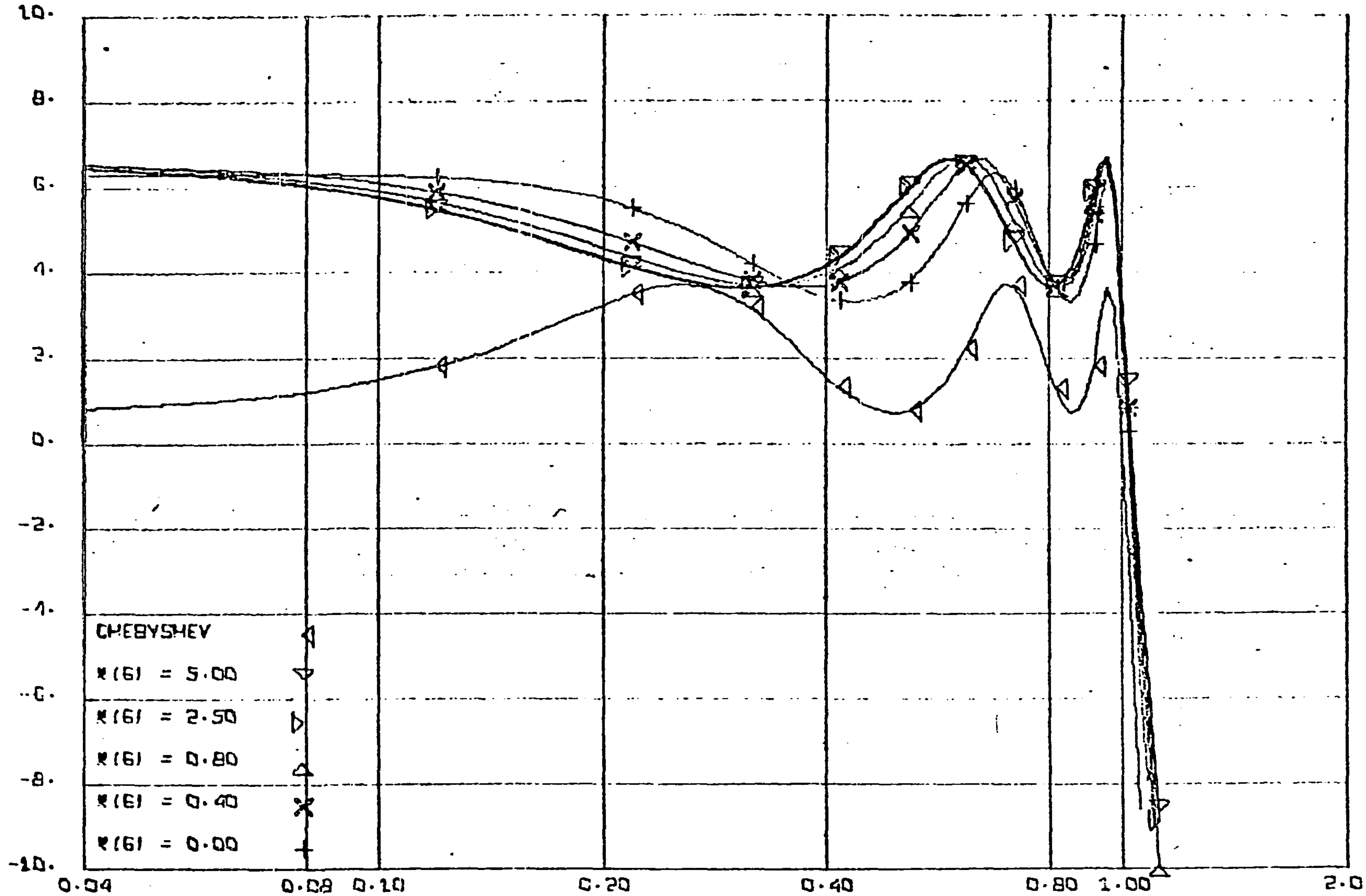


FIGURE 7. FREQUENCY RESPONSE OF ANALOG SYSTEM

Table 3. Network family

x_1	C_1	L_1	C_2	L_2	C_3	L_3
0.0	2.073	1.446	2.273	1.709	1.709	1.306
0.1	2.093	1.418	2.337	1.651	1.760	1.253
0.2	2.126	1.372	2.454	1.544	1.865	1.142
0.3	2.147	1.342	2.543	1.461	1.961	1.029
0.4	2.156	1.325	2.594	1.406	2.030	0.9310
0.5	2.160	1.317	2.621	1.370	2.076	0.8492
0.6	2.161	1.312	2.635	1.346	2.105	0.7800
0.7	2.160	1.310	2.643	1.329	2.121	0.7208
0.8	2.160	1.308	2.645	1.317	2.128	0.6697
0.9	2.159	1.307	2.646	1.308	2.136	0.6252
1.0	2.158	1.306	2.645	1.301	2.127	0.5862
1.5	2.156	1.304	2.640	1.280	2.090	0.4455
2.0	2.154	1.302	2.636	1.270	2.045	0.3590
2.5	2.153	1.302	2.633	1.264	2.008	0.3004
3.0	2.153	1.301	2.631	1.261	1.977	0.2581
3.5	2.153	1.301	2.630	1.258	1.952	0.2262
4.0	2.153	1.301	2.630	1.256	1.931	0.2014
4.5	2.152	1.301	2.628	1.255	1.915	0.1814
5.0	2.152	1.301	2.628	1.254	1.900	0.1650

design is included for comparison purposes.

3. Additional test results

Several additional tests were performed on the networks formed using the extremum method to aid in their evaluation. These tests included sensitivity to polynomial coefficient precision and effects of varying the ripple factor on the component values.

A series of tests were made on the networks to determine the effects of reduced polynomial coefficient precision.

In this series of tests the number of significant figures in the polynomial coefficients was reduced until there was a significant change in the frequency response curves of the resulting networks. The computer generated response curves were used as the basis for comparison. No noticeable change (less than 1%) was noticed until the number of significant figures was reduced to three. For this value the response curves differed noticeably (greater than 5%) from the more precise curves. Therefore, it was decided that four significant figures were satisfactory and this precision was used throughout the remainder of the synthesis and analysis.

A series of tests was performed to determine whether or not the ripple magnitude could be reduced without violating one or more of the inter-component constraints. Table 4 gives the results of one of these tests. For this test the arbitrary parameter was held at 0.5 and the ripple factor varied from 0.5 to 0.0001. The results of this test show that the ripple factor could be reduced to 0.01 before the C_2-L_1 constraint was violated. Tests using other values for the arbitrary parameter showed similar results although the value for r at which the constraint was violated did vary with x_1.

Table 4. Network variations with ripple factor

r	C_1	L_1	C_2	L_2	C_3	L_3
0.5	2.657	1.086	3.209	1.171	2.425	0.9690
0.1	1.650	1.633	1.997	1.589	1.573	0.6311
0.05	1.542	1.680	1.851	1.586	1.399	0.5496
0.01	1.401	1.640	1.640	1.454	1.113	0.4196
0.005	1.354	1.595	1.566	1.379	1.020	0.3793
0.001	1.247	1.469	1.405	1.205	0.8464	0.3070
0.0005	1.200	1.410	1.337	1.135	0.7849	0.2824
0.0001	1.091	1.273	1.188	0.9562	0.6643	0.2356

D. Conclusions

In this particular problem the extremum method permitted the formation of a family of transducer analog networks, all members of which satisfied a variety of specifications imposed on the networks by the nature of the acoustic device and the fabrication process. The application of existing approximation techniques resulted in networks which failed to satisfy these same specifications. The cost of obtaining this one parameter family of networks was the sacrifice of one pass band ripple and the resulting deterioration in the cut-off characteristics.

IV. THE COEFFICIENT METHOD

In this section and the two which follow the coefficient method of forming approximating polynomials is discussed and illustrated. The concepts underlying this method and details of the development of the method based on these concepts are covered in this section. In Section V several examples of polynomials developed using the coefficient method are shown in order to illustrate the versatility of this method. Examples of the application of the coefficient method to problems of network synthesis are included in Section VI.

The major purpose of the coefficient method is to permit a wide variety of functions, irrational as well as rational, to be approximated by polynomials in a form suitable for use in network synthesis. This approximation is to be within specified magnitude limits and over a given range of the independent variable. In addition, it is desired to allow arbitrary parameters to be specified within the method so that multi-parameter families of approximating polynomials can be formed. An additional purpose is to develop a method which accomplishes the above but does not require the solution of a system of nonlinear equations.

A. Characteristics of the Coefficient Method

In the above statement which describes the purpose of the coefficient method, four desirable characteristics are given. These characteristics are:

1. approximation to a wide variety of functions,
2. specified magnitude limits on the approximation,
3. allowable arbitrary parameters,
4. simplified methods of computation.

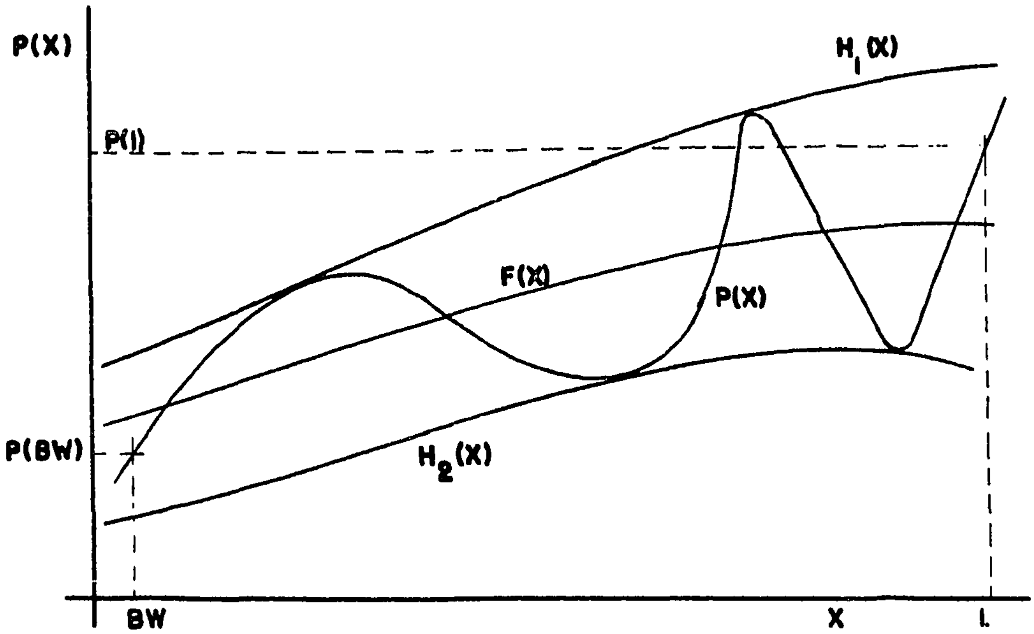
These characteristics are described in detail in the following discussions.

1. Functions to be approximated

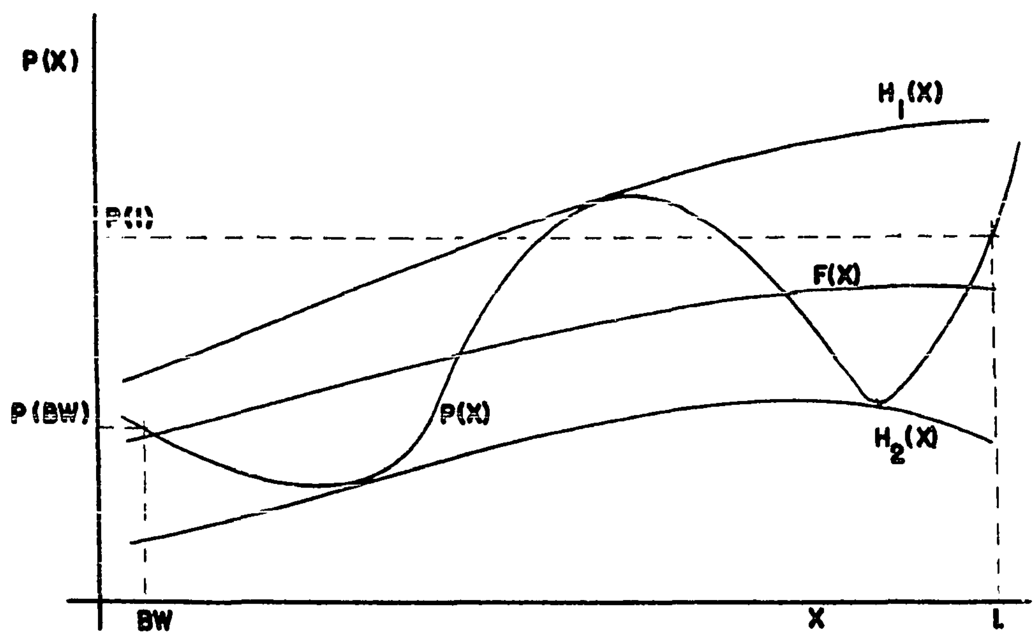
The major purpose of the coefficient method is to permit a wide variety of functions to be approximated by a polynomial in a form suitable for use in network synthesis. For practical purposes the functions of interest must be limited to those which are finite and single-valued over the range of approximation. These functions may be rational or irrational, explicit mathematical forms or piece-wise linear approximations to functions which cannot be specified explicitly.

This approximation is described in Figure 8. In this figure, $f(x)$ is the function to be approximated by the polynomial, $P(x)$. The range of the approximation is from $x = BW$ to $x = 1$. The limits within which the approximation must fall are $H_1(x)$ and $H_2(x)$. This notation is used throughout the discussion.

There are four possible combinations of factors affecting the shape of $P(x)$ in Figure 8. These factors



A. N-M-I EVEN, SLOPE AT X=I. POSITIVE



B. N-M-I ODD, SLOPE AT X=I. POSITIVE

FIGURE 8. POLYNOMIAL APPROXIMATION TO FUNCTION

are the end condition slopes and the number of pass band ripples. Figure 8a indicates the shape of $P(x)$ for an even number of pass band ripples and a positive slope at $x = 1$. In Figure 8b, the number of ripples is odd and there is a positive slope at $x = 1$. If a negative slope at $x = 1$ is desired, $P(x)$ is reflected about $f(x)$ and approaches $H_1(x)$ instead of H_2 at the uppermost approach point.

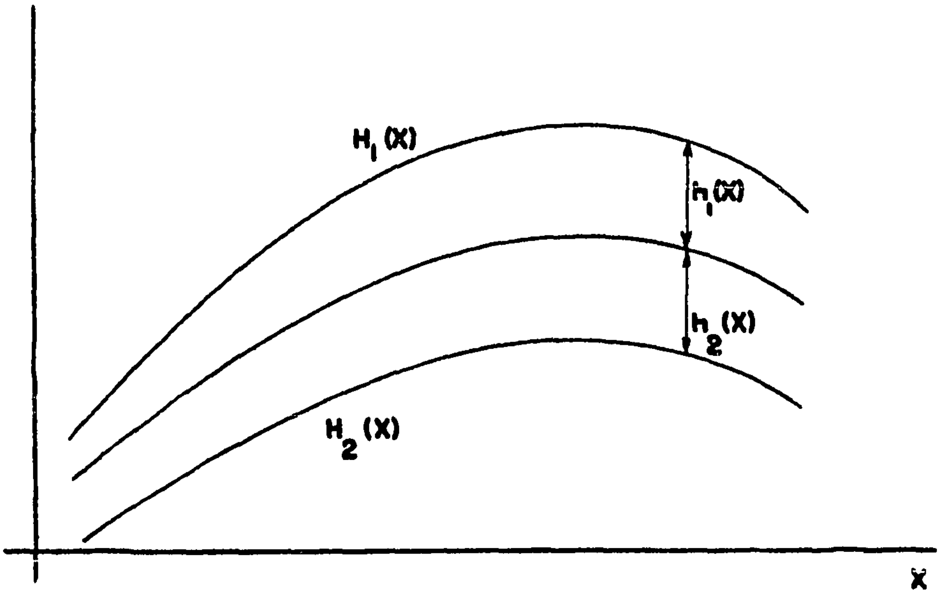
2. Limiting functions

In the coefficient method the limiting functions, $H_1(x)$ and $H_2(x)$, are of more importance than the actual function to be approximated since these functions define the range of variation for the polynomial being developed. These limiting functions are functions of x and are not limited to being equi-spaced about the function $f(x)$.

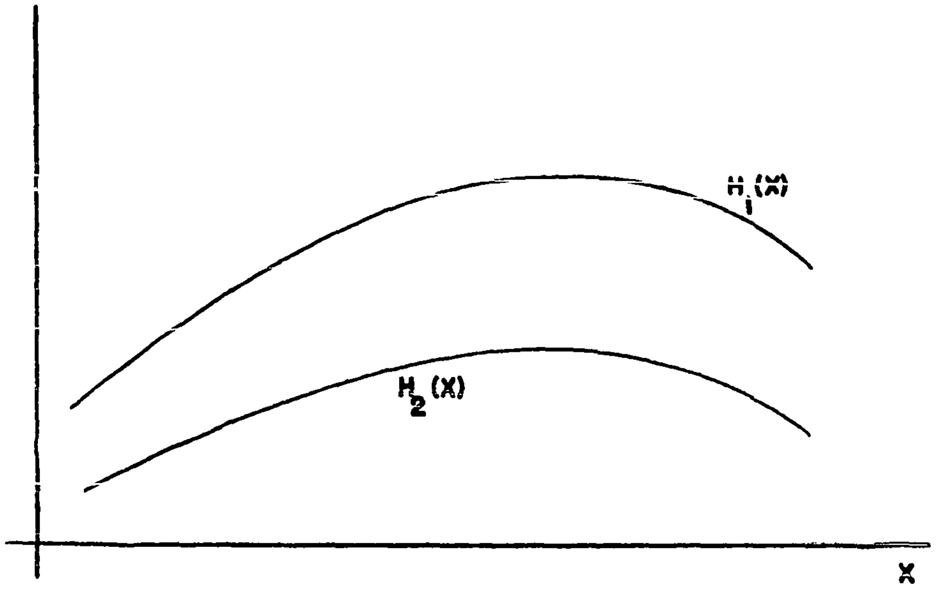
There are two methods of defining these limiting functions for use with the coefficient method. The first involves the use of $f(x)$, the function to be approximated, and is illustrated in Figure 9a. In this procedure, the limits are defined by Equation 15.

$$\begin{aligned} H_1(x) &= f(x) + h_1(x) \\ H_2(x) &= f(x) + h_2(x) \end{aligned} \tag{15}$$

The functions, $h_1(x)$ and $h_2(x)$, may be any finite single-



A. LIMITING FUNCTIONS DEFINED BY VARIATION FROM FUNCTION



B. LIMITING FUNCTIONS DEFINED INDEPENDENTLY OF FUNCTION

FIGURE 9. TYPES OF LIMITING FUNCTION DEFINITIONS

valued functions of x, subject to the restriction $H_1(x) \geq f(x) \geq H_2(x)$.

A more general method of defining the limiting functions is shown in Figure 9b. In this method the limiting functions are defined independently of the function to be approximated. Indeed, no function, $f(x)$, has to be specified. For computational purposes the limiting functions are defined in this manner no matter which method is used in their specification.

3. Arbitrary parameters

The desire for arbitrary parameters within the approximation method is predicated upon the application of this method to a variety of network design problems. Families of networks can be formed using these arbitrary parameters and certain inter-component constraints satisfied by given family members. Also, these parameters may be used to control directly some specific characteristic of the network such as cut-off or d.c. behavior.

In order for the arbitrary parameters to be used to control a variety of characteristics or to satisfy various inter-component constraints, it is necessary to use a variety of parameter definitions. Some of the parameter definitions available using the coefficient method include polynomial coefficients, points through which the plot of the polynomial must pass, zeros of the polynomial and the slope of the polynomial at a point. The d.c. behavior of the resultant network can be specified by control of the constant term of the polynomial, a_0. Network cut-off characteristics may be affected by variation of the coefficient of the highest power of x, a_{2n}, and the slope of the polynomial at a point.

There is one restriction on the choice of the arbitrary parameter value. The value of this parameter must not force the resulting polynomial to cross the limiting functions within the range of the approximation.

4. Improvement in the method of computation

The coefficient method requires computation methods which are much less complicated than those of the extremum method and therefore it can be made faster and more accurate.

The major computations required during one iteration of the coefficient method are the solution of a system of $n + 1$ linear equations and the factoring of a set of polynomials. Both of these computations may be accomplished using standard library procedures. For improved performance, special purpose techniques for performing these operations could be designed, but these are not necessary.

B. Development of the Coefficient Method

The coefficient method is a technique for developing a family of polynomials which fall within two limiting functions, $H_1(x)$ and $H_2(x)$, over the range of x from BW to one and are tangent to thse limits at the points of intersection. The discussion of this method is based on Figure 10. The notation used is defined by this figure. Figure 10 illustrates only one of the four possible configurations discussed in Section IV, A, 1.

The following steps are performed during each iteration in the process of forming a polynomial, P(x), which meets the requirements stated above.

1. A set of estimates to the points of tangency of the approximating polynomial and the limiting functions is obtained. These points, labelled x_i in the diagram, are obtained either from the previous iteration or externally.

2. A polynomial, $P_a(x)$, is formed which intersects $H_1(x)$ and $H_2(x)$ at the x_i and passes through the specified end points. The formation of this polynomial includes the m arbitrary parameters and requires the solution of a set of n + 1 simultaneous equations.

3. The polynomial formed in step 2 is tested for points X_i at which the derivative is equal to that of the correct limiting functions.

4. The set of estimates, x_i, is compared with the set of equal slope points, X_i, to determine if they are coincident within some measure of error. If so, the polynomial $P_a(x)$ is a solution.

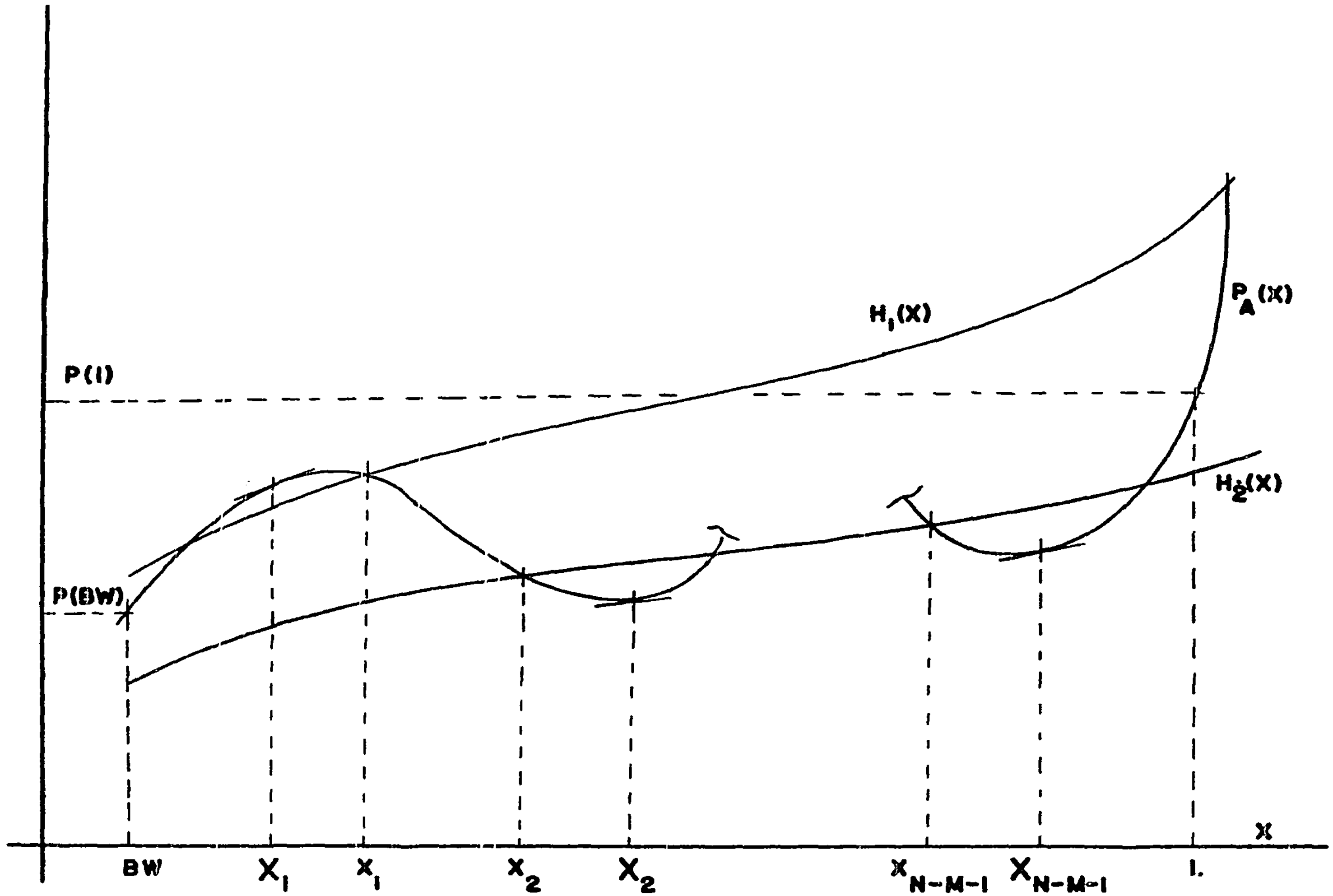


FIGURE 10. GRAPHICAL DESCRIPTION OF COEFFICIENT METHOD

5. If the polynomial is not acceptable as a solution, a new set of x_i is formed based upon the previous x_i and the computed equal slope points, X_i.

6. Steps 2 through 5 are repeated until a satisfactory polynomial is formed. Additional details concerning these steps is given in the following discussion.

1. Data input

There are two types of input required by the coefficient method. The first type includes all necessary data about the characteristics of the resultant polynomial. The second input defines the limiting functions on the approximation.

The polynomial data necessary are the polynomial order, the number of arbitrary parameters, the number of pass band ripples, the dimensions of the pass band, the upper limit in the allowable error measure and the approach point estimates. The order of the polynomial to be used depends upon many factors including the shape of limiting functions, the number of pass band ripples desired, the number of arbitrary parameters needed and the desired precision of the approximation. If the form of the network to be synthesized is specified, this will be a determining factor in the polynomial order also. There is a relation between the number of pass band ripples, the number of arbitrary parameters and the order of the polynomial. If one of these data is varied, a change must be made in the others to maintain this relationship. For an even polynomial of order 2n the number of pass band ripples plus arbitrary parameters must be n-1.

The second type of data required is that necessary to define the limiting functions in a form which can be used in the computational scheme employed. Unless the functions can be expressed as polynomials, it is more convenient to reduce the functions to a data table and use table look-up procedures where necessary. This process allows all functions to be handled using the same basic computational procedures regardless of the method of defining the limits. If the limiting functions are polynomials, the overall computations may be simplified by a special purpose procedure.

2. Initial estimates

The terms symbolized by x_i are the initial estimates of the values of x at which the polynomial is tangent to the limiting functions. This same symbology is used to describe the revised estimates supplied to any other cycle by the previous cycle.

Without previous experience, there is no good way of making an accurate estimate of the x_i, especially if the limiting functions are complicated functions of x. Since the method of computation developed for the coefficient method does not require the solution of a system of nonlinear equations, the choice of the initial estimates is not as critical as in the extremum method. Experience gained in developing the polynomials described in Section V has shown that the coefficient method will converge to a solution for any reasonable estimate of the approach points.

3. <u>Coefficient equation system</u>

The second algorithm of Remes (10, 11) was adapted to provide a method for developing the coefficients of a test polynomial having the form of Equation 16.

$$P(x) = a_1 x^{2n} + a_2 x^{2n-2} + \ldots + a_n x^2 + a_{n+1} \tag{16}$$

The system of n-1 equations given as Equation 17 uses the a_i coefficients as the unknowns and powers of x_i as the equation coefficients. This set of equations is written as though the polynomial intersects the upper limit at x_1 and the lower limit at x_{n-1} as in Figure 10. Two additional

$$\begin{aligned} a_1 x_1^{2n} + a_2 x_1^{2n-2} + \ldots + a_n x_1^2 + a_{n+1} &= H_1(x_1) \\ a_1 x_2^{2n} + a_2 x_2^{2n-2} + \ldots + a_n x_2^2 + a_{n+1} &= H_2(x_2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1 x_{n-1}^{2n} + a_2 x_{n-1}^{2n-2} + \ldots + a_n x_{n-1}^2 + a_{n+1} &= H_2(x_{n-1}) \end{aligned} \tag{17}$$

equations are needed which define the end point conditions. These two equations, given below as Equation 18, plus the (n-1) equations defined by Equation 17, are the (n+1) equations needed to determine the (n+1) coefficients of an

$$
\begin{aligned}
P(BW) &= K_1 \\
P(1) &= K_2
\end{aligned}
\qquad (18)
$$

even $2n^{th}$ order equation. The two constants, K_1 and K_2, in Equation 15 are values of $P(x)$ at the end points of the pass band. These values may be selected arbitrarily subject only to the restrictions $H_1(BW) \leq K_1 \leq H_2(BW)$ and $H_1(1) \leq K_2 \leq H_2(1)$. This set of equations describes the conditions illustrated in Figures 8a and 10 and is used as one possible example of the four possible described in Section IV, A, 1.

Equations 17 and 18 are written as though there are no arbitrary parameters. With each arbitrary parameter used, the number of equations in Equation 17 is reduced by one and replaced by an equation which defines the parameter. If, as is common, one of the coefficients of $P(x)$ is used as a parameter, one equation in the system described by Equation 17 is replaced by $a_k = K_k$ where K_k is an arbitrary constant. Thus the system of equations to be solved consists of m equations defining the parameters, n-m-1 equations for the x_i conditions as in Equation 17 and the two end conditions of Equation 18.

The a_i coefficients are the unknowns in the system of equations formed by combining Equations 17 and 18 and the equations defining the arbitrary parameters. Since the a_i's are the unknowns, the system of equations is linear except for the right hand side of the equations which are functions of x. However, because an estimate of x is used and the limits are defined for all x, these functions appear as known values for any one iteration. The value of these functions will vary with succeeding iterations.

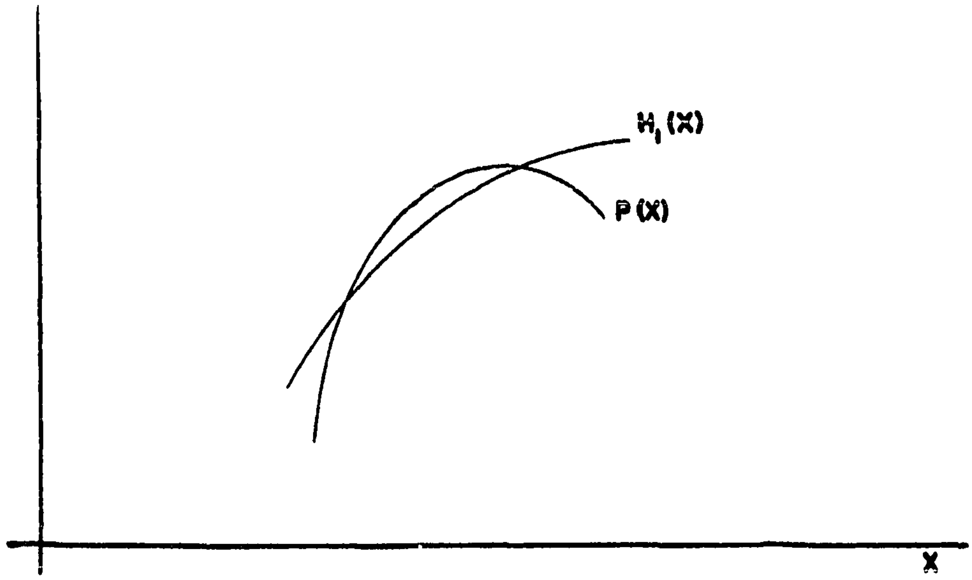
4. Formation of an approximate polynomial

Once the system of equations has been solved for the polynomial coefficients, an approximate polynomial, $P_a(x)$, may be formed. This polynomial, as shown in Figure 10, passes through the estimates of the approach points, x_i, and satisfies the two end conditions. This polynomial, must be examined to determine the points of equal slope, X_i, in order to ascertain whether or not it may be considered a solution.

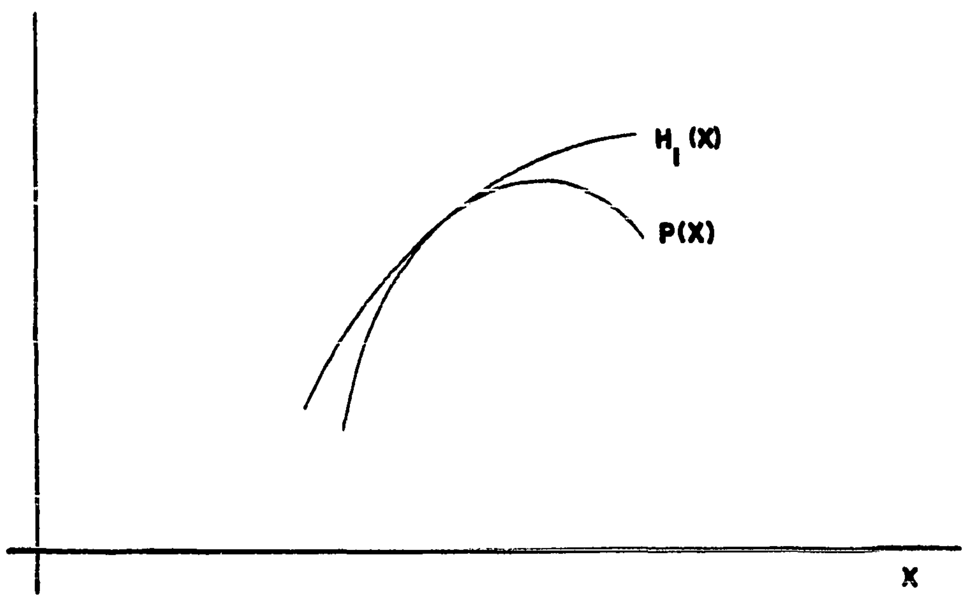
5. Approach conditions

Before $P_a(x)$ can be considered a solution, it and the limiting functions must have equal slopes at their points of intersection. This condition implies that the two curves do not cross as shown in Figure 11a but become tangent as shown in Figure 11b.

The crossing and overshooting shown in Figure 11a is



A. EXTREMUM APPROACH SHOWING OVERSHOOT



B. REDUCTION OF OVERSHOOT USING TANGENTIAL APPROACH

FIGURE II. COMPARISON OF APPROACH CONDITIONS due to the slope of $P_a(x)$ being larger than the slope of $H(x)$ as $P_a(x)$ approaches $H(x)$. This is the condition which requires a tangential approach to the limits rather than the extrema approach used in the extremum method. The curves of Figure 11a satisfy the extremum approach in that the second intersection occurs at the extremum of $P(x)$, but overshooting occurs. The tangential approach of Figure 11b ensures that overshooting will not take place unless there are abrupt changes in the slope of $H(x)$.

Since there are two methods of defining the limiting conditions, explicit mathematical expression or piece-wise linear approximation, two methods for testing the approach conditions are required. The first of these is applicable whenever the limiting functions are expressed as polynomials and the second is used whenever the limiting functions must be expressed as transcendental functions or as piece-wise linear approximations.

For limiting functions which can be expressed as polynomials in x, the test of approach conditions requires the roots of Equation 19 to be found. If $H_1(x) - H_2(x)$ is a constant for all x over the range of the approximation,

$$P_a'(x) - H_1'(x) = 0 \tag{19a}$$

$$P_a'(x) - H_2'(x) = 0 \tag{19b}$$

only one of these equations is needed. Since factoring the two equations will give a total number of roots which is twice the number required, the roots must be arranged in numerical order and selected to ensure that the limits are approached alternately. Using Figure 10 as an example, the first root is selected from the set found by solving for the roots of Equation 19a. The second root is selected from the roots of Equation 19b and so on until the appropriate number of roots has been selected. This set of roots is the set of points of equal slopes designated X_1 in Figure 10. If $H_1(x)$ and $H_2(x)$ are not polynomials, Equations 19a and 19b are not easily factored and piece-wise linear approximations must be used.

In the event that the limiting functions must be approximated in a piece-wise linear fashion, the test for equal slope points becomes more complicated. Equation 19 must be replaced by a system of equations each having the form given in Equation 20. In this equation the x_1's represent

$$P_a'(x) - SL(x_1) = 0 \qquad (20)$$

the approach point estimates and it is necessary to find the roots of as many equations of this form as there are approach points to be tested. The term, $SL(x_1)$ is the slope of the piece-wise linear approximation in the vicinity of x_1. In this case, as in the one above, there are more roots than are needed and appropriate values for the x_1's must be selected.

There are many methods which could be used to select an appropriate set of roots to use as the points of equal slope. One method of selecting the proper root from each of the root sets is illustrated by Figure 12. For each root set a range of values is specified and the first root falling within that range is selected. The method of specifying this range using midpoints as shown in the figure has proved successful for most of the polynomials developed. In the cases for which the midpoint method has failed, it has been found that a tightening of the range of the roots nearest unity has prevented oscillation about a solution and a final solution has been obtained.

6. Test of the approach points

At this point in the computation cycle, there are two sets of values for x. The first is the original set of estimates used to form Equation 17 and to determine the coefficients of the test polynomial. The second set is that obtained from the approximate polynomial in the manner described in the previous section and which satisfies the approach condition. The difference between these two sets is used as a measure of convergence.

The test for convergence is based on the sum of the squares of the differences between the two sets of estimates. When this sum of squares is less than some specified maximum,

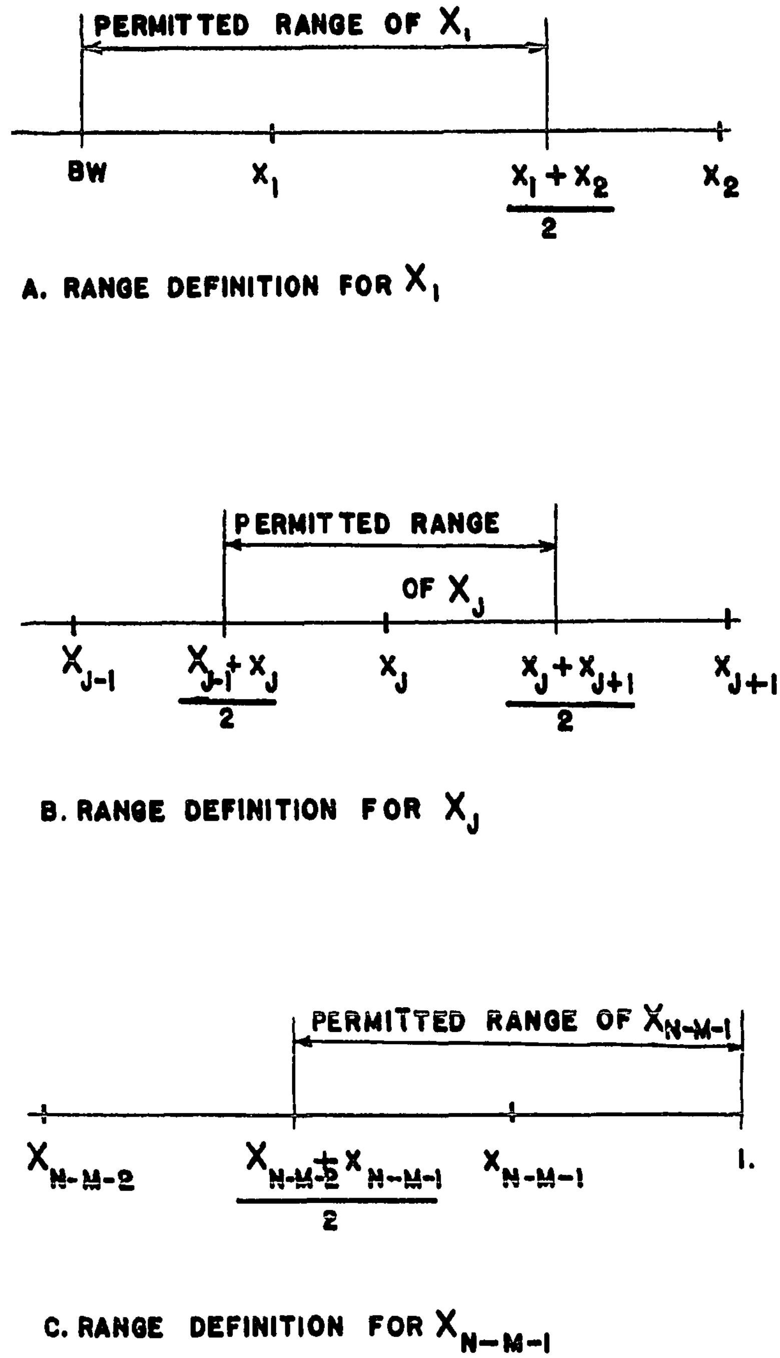


FIGURE 12. MIDPOINT METHOD OF ROOT SELECTION

the test polynomial is considered to be a solution to the approximation problem. Through experience it has been found that, with proper choices of the value of the arbitrary parameter, this maximum value should be 10^{-6} or less. For some choices of the value of the arbitrary parameter, it has been found that the polynomials are sensitive to changes in the approach points. Therefore, the maximum value of the measure of convergence must be made smaller than the value of 10^{-6} suggested.

If the measure of convergence is greater than the prescribed maximum, the set of x's which do satisfy the approach conditions can be used as the next set of approach point estimates for the next cycle of calculations. This choice permits rapid convergence to a solution but also increases the tendency of the method to diverge. To reduce the possibility of divergence, a revised set of estimates is used whenever the most recently computed measure of convergence is greater than the previous minimum. The method of revision chosen was to use the midpoint between the previous best estimates and the present estimates as the input to the next cycle. There are many other possible revision techniques, but this one has been found to prevent divergence and permit convergence to a solution within a reasonable number of iterations.

C. Properties of the Coefficient Method

The coefficient method as derived in the previous sections has all the desired characteristics listed in Section IV, A. This method permits the formation of a multi-parameter family of polynomials all of which are approximations to the same function. The limits on the allowed deviation from this function may be arbitrary functions of x. The function to be approximated and the limits within which the approximation must remain may be defined either as explicit mathematical functions or as data tables.

1. Control variables

There are properties of the approximation other than the arbitrary parameters which can be easily varied as input parameters. These properties include the magnitude of the variation about the function, the band width of the approximation, and the order of the polynomial.

The magnitude of the allowed variation from the desired function is accomplished by varying the limiting functions. This is done either by redefining some explicit function or by providing a new data table to describe the functions.

The band width, or range of x over which the approximation is to be made, is changed by varying the lower cut-off frequency. This band width parameter is symbolized as BW in Figures 8 and 10. Since the upper cut-off value has been normalized to unity, this parameter is defined as some fraction of this upper range and as such will always be less than one.

The order of the polynomial to be formed is controlled by an input quantity. When the order of the polynomial is changed, either the number of approach points within the pass band must be changed or the program must be redesigned to accommodate an increase in the number of arbitrary parameters.

Another characteristic of the method which may be considered as variable is the method of approach to the limits. All previous discussion has dealt with the method using the tangential approach. In theory, this approach condition may be changed, although, in practice, it will seldom be necessary to do so. One change which may be of practical importance deals with the idea of closest approach instead of actual equality. If this is desirable, the equality signs of Equation 14 must be changed to inequality signs and the problems in computation become much more severe. An alternative to this change is the possibility of varying the limits to within the original set and attempting to form the more precise approximation. If the limiting functions are properly defined, this problem should never arise in practice.

2. Comparison with the extremum method

The extremum method was designed to be a special purpose approximation technique to solve a particular problem. For this purpose it has proved to be satisfactory. The coefficient method was designed to be a versatile multipurpose approximation technique which could be applied to a wide range of problems. So far as can be determined from present experience, it also performs satisfactorily.

The coefficient method permits the formation of multiparameter families of polynomials which approximate various functions to within limiting functions which may vary with the independent variable. The extremum method is limited to form multi-parameter families of polynomials which approximate a constant in an equi-ripple manner. It is possible that the extremum method could be revised such that the equi-ripple characteristic is not a requirement, but since the coefficient method is available, it is preferable to use the coefficient method and to consider the extremum method as a special purpose technique only.

Since it contains a band width control factor which is variable, the coefficient method can be used in the design of band pass filter networks. The extremum method was designed to be used in the design of low pass filter networks. Once again, it is possible to revise the extremum method to enable it to be used for low pass and band pass design, but the coefficient method makes this revision unnecessary.

The two methods were derived using similar hypotheses. Despite this similarity in origin, the calculations involved in the two methods are quite different. Both methods require the solution of a system of equations which includes both linear and nonlinear equations. In the extremum method, the degree of nonlinearity is very high and most of the problems in the computation are due to this nonlinearity. In the coefficient method, the degree on nonlinearity is very low and no solutions of system of nonlinear equations are actually required.

Improved methods of factoring polynomials and solving systems of nonlinear equations would increase the usefulness of both methods by significantly reducing the time required to reach a solution.

3. Comparison with other methods

Of the methods of approximation outlined in the introduction only the method developed by D. S. Humpherys (8) is similar enough to the coefficient method to permit meaningful comparison. The other methods mentioned are limited in their application by lack of flexibility.

The method proposed by Humpherys is designed to produce rational function approximations of a polynomial in an equi-ripple fashion. The use of a rational function to accomplish the approximation is one major difference between Humpherys' method and the coefficient method, which uses a polynomial for the approximation. There are advantages associated with either specification. Use of polynomials allows the numerator and denominator polynomials of a network function to be specified separately. The entire function may be specified as a unit if a rational function approximation is used. The particular application is the determining factor in the decision of which specification is to be used.

Humpherys' method is more general than other rational function approximation techniques. It does permit the approximation of a variety of polynomials in an equi-ripple manner with fewer than the maximum number of pass band ripples. Therefore, it permits the use of arbitrary parameters similar to the coefficient method. In the method as described, these arbitrary parameters are limited to imaginary zeros of either the numerator or denominator polynomials. It is possible that the method could be revised to allow other definitions of the arbitrary parameters. However, if this were done, the simplicity of the computations would be lost.

The major advantage of Humpherys' method is the fact that only simple linear equations must be solved once a proper set of conditions has been established. This simplicity is a result of three limitations on the method. The first limitation, as stated above, is the restriction on the choice of the arbitrary parameters. The second limitation involves the approach condition. Humpherys' method assumes that an extremum approach will be adequate for the functions to be approximated. It has been shown in this thesis that this is not necessarily true. The third limitation is that the function to be approximated must be expressed as a polynomial. Any of these limitations may be removed by proper revision of the method, but this is accomplished at the cost of increased complexity of the required computations. The coefficient method was designed to be free of these limitations and thus requires much more in the way of computation than does the method of Humpherys', however it is more versatile.

4. Problems associated with the coefficient method

The major problems associated with the coefficient method are those due to difficulties in the computational processes. These difficulties result in reduced accuracy and increased computation times. Unlike the extremum method, there is no requirement that a system of nonlinear equations be solved. Instead, the coefficient method requires that, during each iteration, a number of polynomials equal to the number of points of approach within the pass band be factored. An estimated 80 percent of the time required for each cycle is used in these factoring processes. This time is reduced appreciably if the limiting functions can be expressed as polynomials since for that case, only two polynomials must be factored for each cycle. At present, industry supplied programs are used for factoring these polynomials. These are iterative methods and may, in some cases, diverge or give inaccurate answers. For examples in which the limiting functions have large slopes these inaccuracies are often large enough to prevent the method from converging to a solution.

One possible solution to the above problem is the design of a special purpose program to factor the polynomials. The programs now used are general purpose methods used to find all the roots of the polynomials. Since only the positive real roots falling within a given range are of interest, a program could be designed to find only these roots. Such a program could prove to be faster and more accurate than the general purpose ones now used.

A problem arises whenever the limiting functions have abrupt changes in slope. It is possible that in the vicinity of such changes the approximating polynomial may overshoot the limits (See Figure 23, Section V, B for an example). There are two possible solutions for this problem. The first would be to include within the program additional calculations which would test every point along the curve for such overshooting. The second possible solution would be a redefinition of the approach conditions to include higher order derivatives. In either case, the increase in computation may be impractical and will not necessarily ensure that such limit crossings do not take place.

Attempts to form a general program met with a general lack of success. Because of the large variety of possible arbitrary parameter definitions, no one program can include all choices. It is felt that the improved accuracy and reduced computation time possible with special purpose programs make such programs more desirable than an all-purpose program which attempts to do everything for everybody. Once the basic ideas are understood and several fundamental decisions have been made for the particular problem, the design of a specific program is not difficult.

D. General Remarks

The coefficient method developed in this section of the thesis allows the formation of multi-parameter families of polynomials. Within these families, all polynomials have graphs which fall between two limiting functions. The number of parameters used and their definitions may be chosen at will and thus may be used to shape specific portions of the network frequency response. The limiting functions to be used are not restricted to explicit mathematical expressions, but may be defined as data tables, thus permitting an almost unlimited choice of limiting functions.

Several problem areas do exist in the actual calculations involved in the coefficient method, but these do not detract from the ideas behind the method. These problems are largely results of programming difficulties. As improved methods of analysis and calculations are developed, the effects of these problem areas will be reduced.

Unlike the extremum method, the coefficient method was designed to be as general as possible and not as a solution to one particular problem. For this reason no complete design, similar to that of Section III has been carried out using the coefficient method. At the time of this writing, it has been used in the design of a more sophisticated transducer than that of Section III, but the design is not complete.

V. EFFECTS OF CONTROL FACTORS

There are several control factors within the coefficient method, exclusive of the arbitrary parameter, which affect either the convergence to a solution or the properties of the approximation polynomial being formed. This section includes a discussion of these factors and illustrates the effects on the polynomials being formed when these factors are changed in value.

The first factors of interest are those which affect the tendency of the method to converge to a solution. In many cases these factors have no direct effect on the approximation, although they may permit formation of a more precise approximation. This precision is usually indicated by the behavior of the polynomial in the vicinity of the approach points. The more precise the approximation, the closer the polynomial and the limiting function approach true tangential intersection. These factors include the maximum value of the measure of convergence, the value of the particular arbitrary parameters being used and the approach point estimates.

The second set of factors of interest includes those factors which directly affect the shape of the approximating polynomials and the range of the approximation. These include the pass band width factor, the order of the polynomial, the limiting functions and the number of ripples within the pass band. These factors are specified independently of the arbitrary parameter definitions or their numerical values and define the approximation problem.

A. Factors Affecting Convergence

The factors affecting the convergence of the coefficient method which are of interest are those which have an effect on the convergence properties but are not part of the approximation definition.

1. Maximum measure of convergence

The measure of convergence within the coefficient method has been defined as the sum of the squared differences between the original estimate set used to initiate any one iteration and the set of improved estimates resulting from that iteration. This sum must be less than some specified maximum measure of convergence before the polynomial formed in that iteration can be considered to be a solution.

The selection of a value to be used for this maximum is dependent upon several considerations. Among the points to be considered are the shape of the limiting functions, the number of pass band ripples and the values of the arbitrary parameter.

The shape of the limiting functions has a decided affect upon the convergence and the precision of the approximation for a given maximum measure of convergence.

For a required precision, the maximum measure of convergence must be reduced in problems containing limits which have large slopes to avoid significant errors in the area of maximum slope.

The number of pass band ripples affects the necessary value of the maximum measure of convergence because of the definition of this quantity as a sum of terms. For a given number of pass band ripples, the average precision of each approach point can be defined as the maximum measure of convergence divided by the number of pass band ripples. Therefore, the solution to an approximation with many pass band ripples must be more precise than that for one with fewer ripples using the same measure of convergence because the average allowed deviation is smaller. For this reason the allowed maximum measure of convergence must be based upon the number of pass band ripples. The larger the number of ripples, the larger may be the maximum measure of convergence.

Another factor affecting the selection of the maximum measure of convergence is the value of the arbitrary parameter. As is discussed in more detail in the following section, this value has a direct effect on the sensitivity of the polynomial to changes in the approach points and must be considered in the choice of a proper convergence factor.

The number of iterations required for solution is strongly dependent upon the maximum allowable value for the measure of convergence. For well behaved functions a value of 10^{-6} has proved to be satisfactory in permitting rapid convergence to a solution and a precision to within 1 percent. If the value is reduced by a factor of 10 the number of iterations required for solution will more than double. As the value is reduced further, the number of iterations required for a solution increases rapidly.

The value for this maximum measure of convergence used in the examples of this section was 10^{-6}. This value was chosen as a compromise between adequate precision and rapid convergence to a solution. As can be seen on several of the figures, this value is not always low enough for acceptable precision. In some cases of rapidly converging iterations the actual measure of convergence was as low as 10^{-8}.

2. Optimum range of parameter values

If the value of the arbitrary parameters is chosen at random, the convergence to a solution may be affected. It is possible to select values for which the method will not converge within some finite number of limits. Attempts to form a polynomial to approximate x^3 failed to converge within 200 iterations for one choice of the value of the arbitrary parameter. When this value was changed, convergence to a satisfactory solution was obtained within 20 iterations. This same effect has been noted in many of the examples solved while testing the method.

Experience has shown that for most problems there is an optimum range for the value of the arbitrary parameters. For values within this range, convergence is obtained rapidly and the approximation does not overshoot the limits or fail to intersect the limits. Polynomials formed using parameter values outside of this optimum range are sensitive to changes in approach points and do not become tangent to the limiting curves unless the maximum measure of convergence is reduced by at least an additional factor of 10 below that needed for approximations using values within the optimum range.

Several of the examples which follow illustrate this increased sensitivity to changes in approach points. This is usually manifested by failure of the curves to become tangent to the limit curves. This failure can be overcome by either changing the value of the arbitrary parameter or reducing the maximum allowable measure of convergence.

3. Approach point estimates

The convergence to a solution of the coefficient method is not strongly dependent upon the initial estimates of the approach points. This freedom from the requirement for an accurate set of estimates is primarily due to the fact that only linear equations are used in the iterative process. Experience using the method has indicated that the process will converge for a variety of estimates within the pass band.

For most of the examples which follow, the same set of approach point estimates was used. The number of iterations required for convergence varied with the problem being solved from a maximum of 39 to a minimum of 8. For examples in which estimates were obtained from previous solutions convergence to a solution occurred within 10 iterations. If no previous knowledge of accurate approach point estimates is available, it has been found that a set of points roughly equally spaced between BW and one serves as a satisfactory set.

B. Factors Affecting Polynomial Shape

Within the coefficient method there are factors which may be varied to control the shape of the approximating polynomial. These factors are used to define the approximation problem and are chosen independently of both the arbitrary parameters and the factors which affect the convergence conditions. These control factors include the band width of the approximation, the spacing between the limits, the order of the polynomial, the approach conditions and the method of defining the limiting functions.

Each of these factors is discussed and illustrated in the following sections. In each case only the factor being discussed is varied, although in several cases a family of curves is shown due to variations in the arbitrary parameter.

1. Band width

For purposes of this thesis the term "band width" is defined to mean that portion of the total frequency reponse within which the response shaping is to be performed. In many of the curves shown there is no sharp cut-off of the high or low frequency response, but rather a gradual sloping away from that part of the response which was to be shaped by the polynomial.

The polynomials formed for this example are one-parameter approximations to x within $\pm$ 0.33x. The co-efficient of x^{14} was chosen as the arbitrary parameter and set equal to 600. The tangential approach condition was used.

Figure 13 shows the curves of the polynomials which result when the width of the pass band is varied. The values of the band width factor, BW, are 0.05, 0.10, 0.15, and 0.20. As the width of the pass band is decreased, the cut-off at low frequencies becomes more apparent. For low values of the band width parameter, there is little decrease in response below the lower cut-off frequency.

Even though the measures of convergence are similar for the four polynomials, there are obvious errors in the

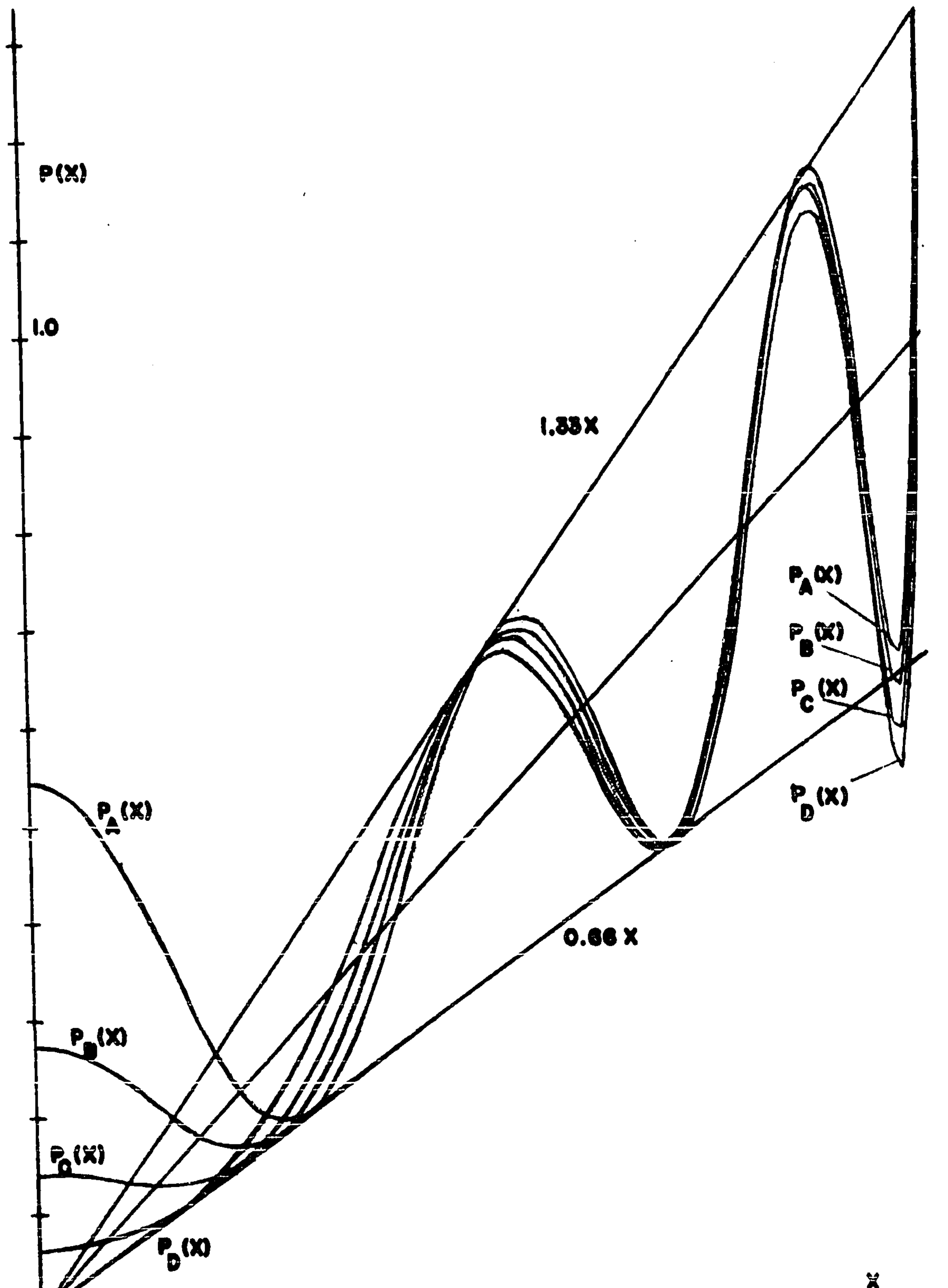


FIGURE 13. BAND WIDTH FACTOR EFFECTS

approximation as the band width parameter is increased. This indicates an increased sensitivity to changes in the points of approach to the limits. This increased sensitivity may be reduced and the errors in the approximation eliminated by proper choice of the value of the arbitrary parameter. In each of the polynomials plotted, the value of the arbitrary parameter was held to 600.

a. BW = 0.20 $\quad A_{2n} = 600 \quad$ ripple factor = 0.33

$$P_a(x) = 600x^{14} - 1615x^{12} + 1437x^{10} - 339.8x^8 - 157.8x^6 + 86.47x^4 - 10.17x^2 + 0.5454$$

b. BW = 0.15 $\quad A_{2n} = 600 \quad$ ripple factor = 0.33

$$P_b(x) = 600x^{14} - 1650x^{12} + 1560x^{10} - 510.8x^8 - 42.48x^6 + 47.74x^4 - 4.378x^2 + 0.2741$$

c. BW = 0.10 $\quad A_{2n} = 600 \quad$ ripple factor = 0.33

$$P_c(x) = 600x^{14} - 1637x^{12} + 1643x^{10} - 623.3x^8 - 31.81x^6 + 23.69x^4 - 1.011x^2 + 0.1408$$

d. BW = 0.05 $\quad A_{2n} = 600 \quad$ ripple factor = 0.33

$$P_d(x) = 600x^{14} - 1687x^{12} + 1697x^{10} - 706.5x^8 + 91.05x^6 + 4.048x^4 + 1.599x^2 + 0.0625$$

2. Spacing between limits

In some approximation problems it may be desirable to be able to vary the spacing between the limiting functions without actually varying the functions themselves. This is especially true in those cases in which the limits are defined by explicit mathematical expressions.

In the approximation problems of the previous example, the limiting functions were defined as;

$$H_1(x) = 1.33x$$

$$H_2(x) = 0.67x$$

For purposes of varying the ripple (i.e., changing the spacing between the limiting functions) the limiting functions are defined by Equations 21. As the ripple factor, R, is varied, the limiting functions change as

$$\begin{aligned} H_1(x) &= (1 + R)x \\ H_2(x) &= (1 - R)x \end{aligned} \qquad (21)$$

shown in Figure 14. The polynomials plotted in this figure are;

a. $R = 0.25 \quad A_{2n} = 600 \quad BW = 0.1$

$$P_a(x) = 600x^{14} - 1784x^{12} + 1960x^{10} - 956.8x^8 + 188.9x^6 - 8.129x^4 + 1.149x^2 + 0.1143$$

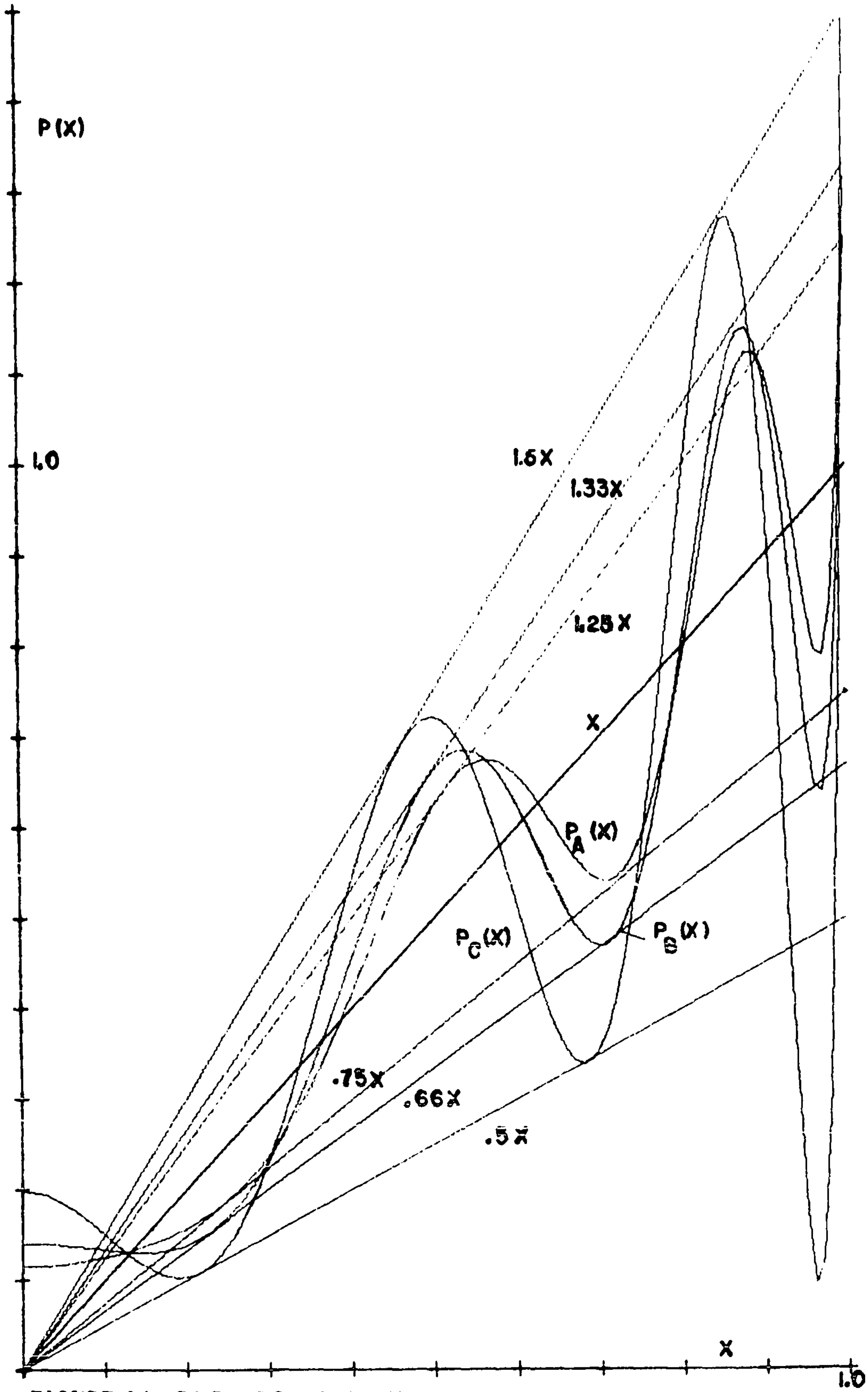


FIGURE 14. EFFECTS OF LIMIT SPACING CHANGES

b. $R = 0.33 \quad A_{2n} = 600 \quad BW = 0.1$

$$P_b(x) = 600x^{14} - 1673x^{12} + 1644x^{10} - 624.0x^8 + 32.14x^6 + 23.65x^4 - 1.007x^2 + 0.1408$$

c. $R = 0.50 \quad A_{2n} = 600 \quad BW = 0.1$

$$P_c(x) = 600x^{14} - 1347x^{12} + 755.8x^{10} + 259.4x^8 - 358.7x^6 + 97.61x^4 - 5.729x^2 + 0.1979$$

As in the previous example, there is an apparent change in polynomial sensitivity to changes in points of approach. A better choice of values for the arbitrary parameter will reduce the error caused by this increased sensitivity.

The polynomial

$$P(x) = 6x^{14} - 16.2x^{12} + 16.5x^{10} - 8.11x^8 + 2.78x^6 + 0.114x^4 - 0.000843x^2 + 0.0000573$$

is a one parameter approximation to x^5 over the range $x = 0.1$ to $x = 1.0$ using a ripple factor $R = 0.01$. No problems regarding convergence were encountered once the value of A_{2n} was reduced in values below 20. This approximation is illustrated in Figure 15. No attempt has been made to approximate a function using a smaller ripple factor.

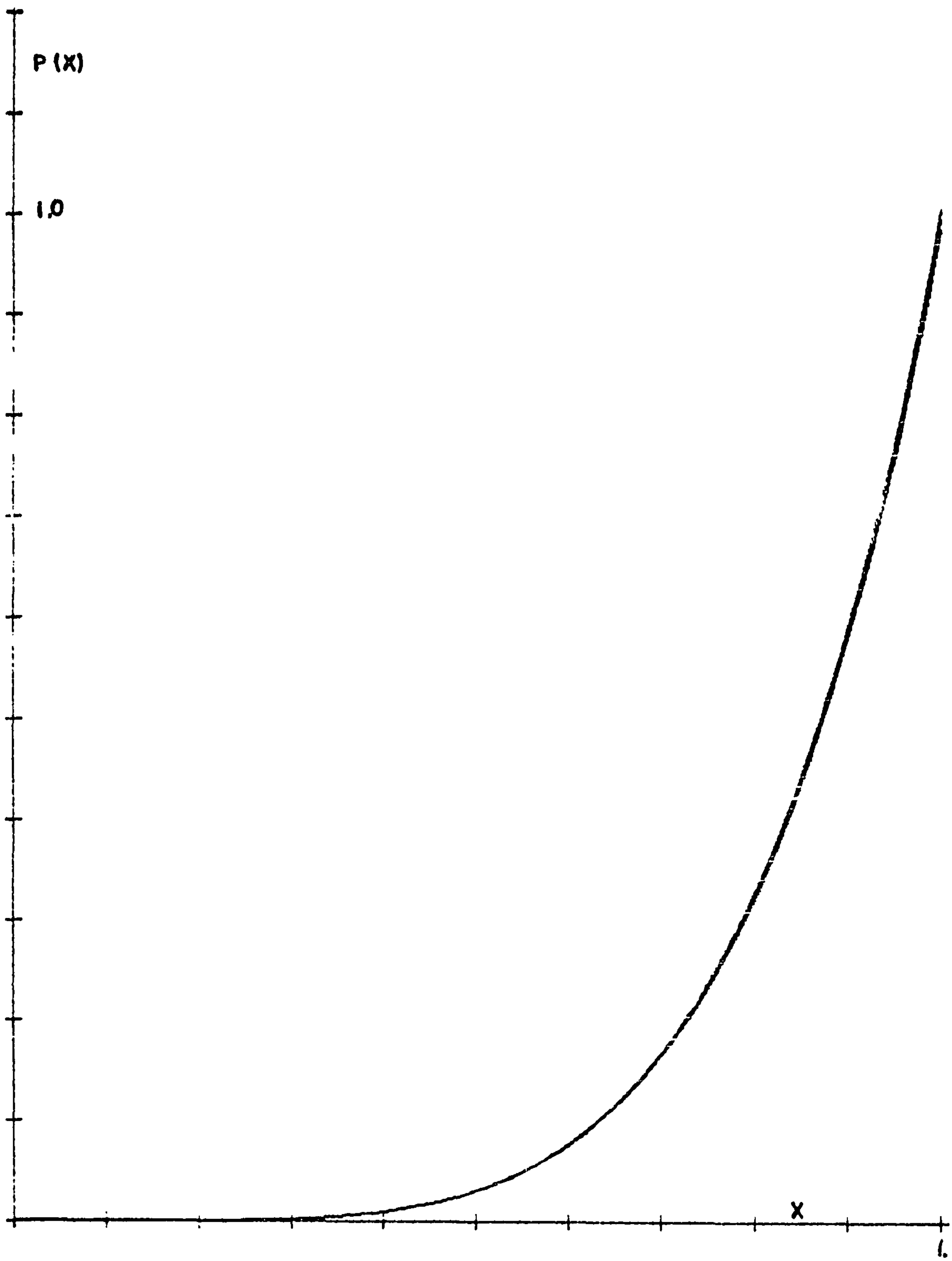


FIGURE 15. 1% APPROXIMATION TO X^5

3. <u>Polynomial order and pass band ripples</u>

One of the more important properties of the coefficient method allows the order of the polynomial being formed to be varied without affecting the remainder of the approximation definition. The only limitation on this variation is dictated by the interrelation of the polynomial order, the number of pass band ripples and the number of arbitrary parameters. Unless a change in the number of arbitrary parameters is desired, the number of pass band ripples must increase as the order of the polynomial increases.

One important consideration in the choice of the order of the approximating polynomial must be in the shape of the function being approximated. If the approximating polynomial has its maximum slope less than the slope of the function being approximated in the range of the approximation, there is a minimum spacing between limits which is dictated by this condition. Normally, the approximating polynomial will be chosen to have order high enough so this condition should never arise in practice.

The preceding examples have used polynomials of fourteenth order and one arbitrary parameter thus requiring five pass band approach points. The polynomials plotted in Figure 16 were developed using the same program that was used for the other polynomials with one change. For this family of polynomials plotted the order of the polynomials

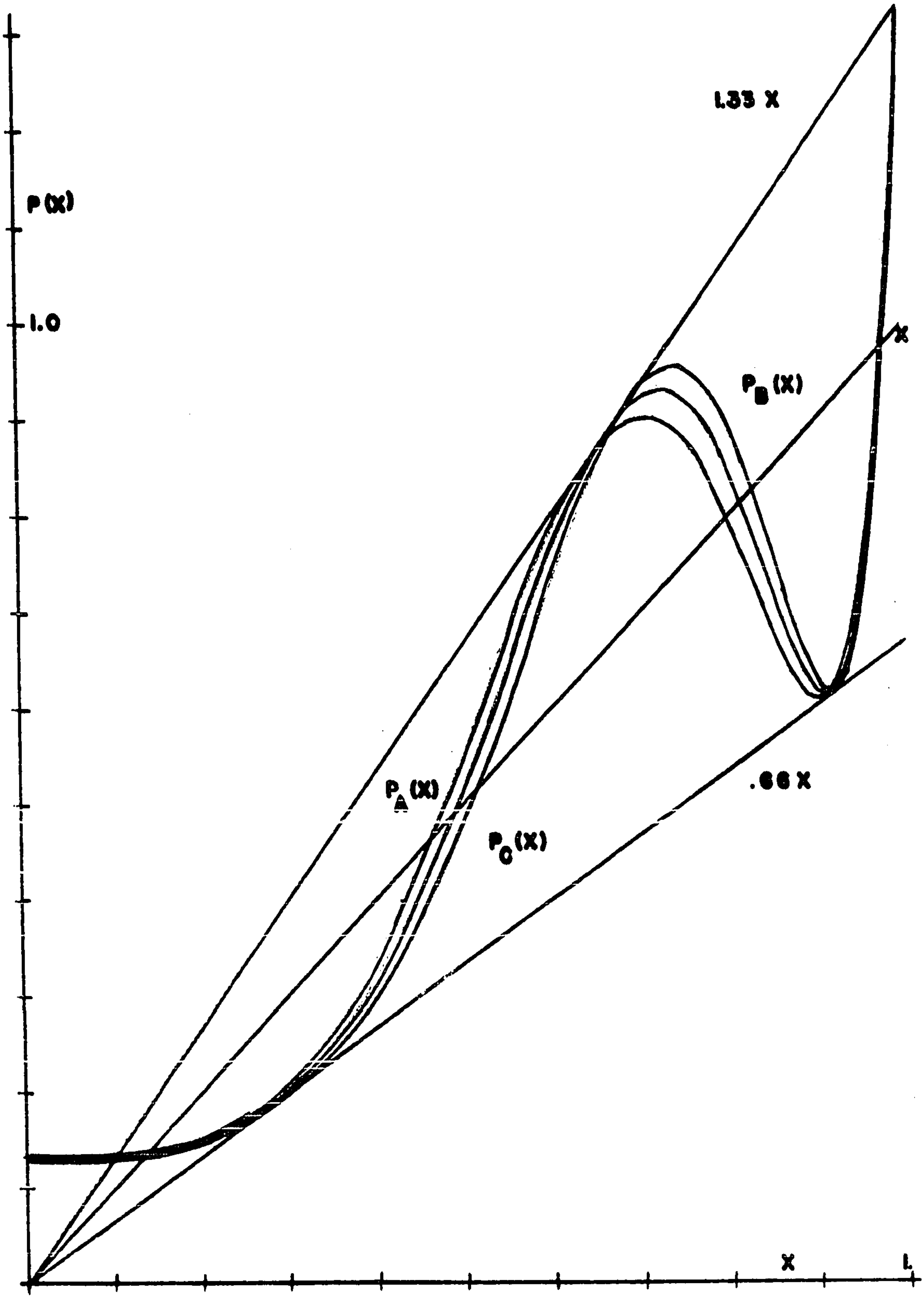


FIGURE 16. APPROXIMATION TO X USING TENTH ORDER POLYNOMIALS

was reduced to ten. This reduced to three the number of points at which the polynomial approaches the limiting function. The polynomials plotted are;

$$P_a(x) = 10x^{10} - 2.727x^8 - 17.32x^6 + 11.31x^4 - 0.0674x^2 + 0.1326$$

$$P_b(x) = 20x^{10} - 26.27x^8 + 0.9353x^6 + 6.286x^4 + 0.2496x^2 + 0.1299$$

$$P_c(x) = 30x^{10} - 50.34x^8 + 20.19x^6 + 0.7169x^4 + 0.6335x^2 + 0.1266$$

The value of the arbitrary parameter was reduced to obtain convergence, and allowed to vary to produce a family of polynomials.

The polynomial

$$P(x) = 100x^{18} - 5800x^{16} - 2305x^{14} + 35891x^{12} + 12319x^8 - 28548x^{10} - 2804x^6 + 297x^4 - 9.366x^2 + 0.1974$$

was developed using the same program as the other polynomials of this section. The order of this polynomial was raised to eighteen and the number of pass band approach points to seven. The band width factor was 0.1 and the ripple factor

0.33. This polynomial is plotted in Figure 17.

The value used as the arbitrary parameter is not within the optimum range, as is obvious by noting the errors at high values of x on the plot. From experience with the tenth and fourteenth order polynomials, it appears that the value of the arbitrary parameter should be increased by at least a power of ten for improved results.

As should be expected, these polynomials were the first to indicate convergence times which differed significantly from the other examples. The time and the number of iterations required for convergence is directly related to both the order of the polynomial and the number of approach points within the pass band. The reason for the dependency on the polynomial order is the number of operations required to solve a system of equations and to factor the polynomials. The number of operations required by both these calculations rises exponentially with increasing polynomial order.

4. Approach conditions

Although the approach condition is not properly a variable factor in any one approximation definition, its definition does affect the shape of the curves and the precision of the approximation. The two approach condition definitions which must be considered are the tangential approach and the extremum approach. The tangential approach is the one used in all examples of Sections V

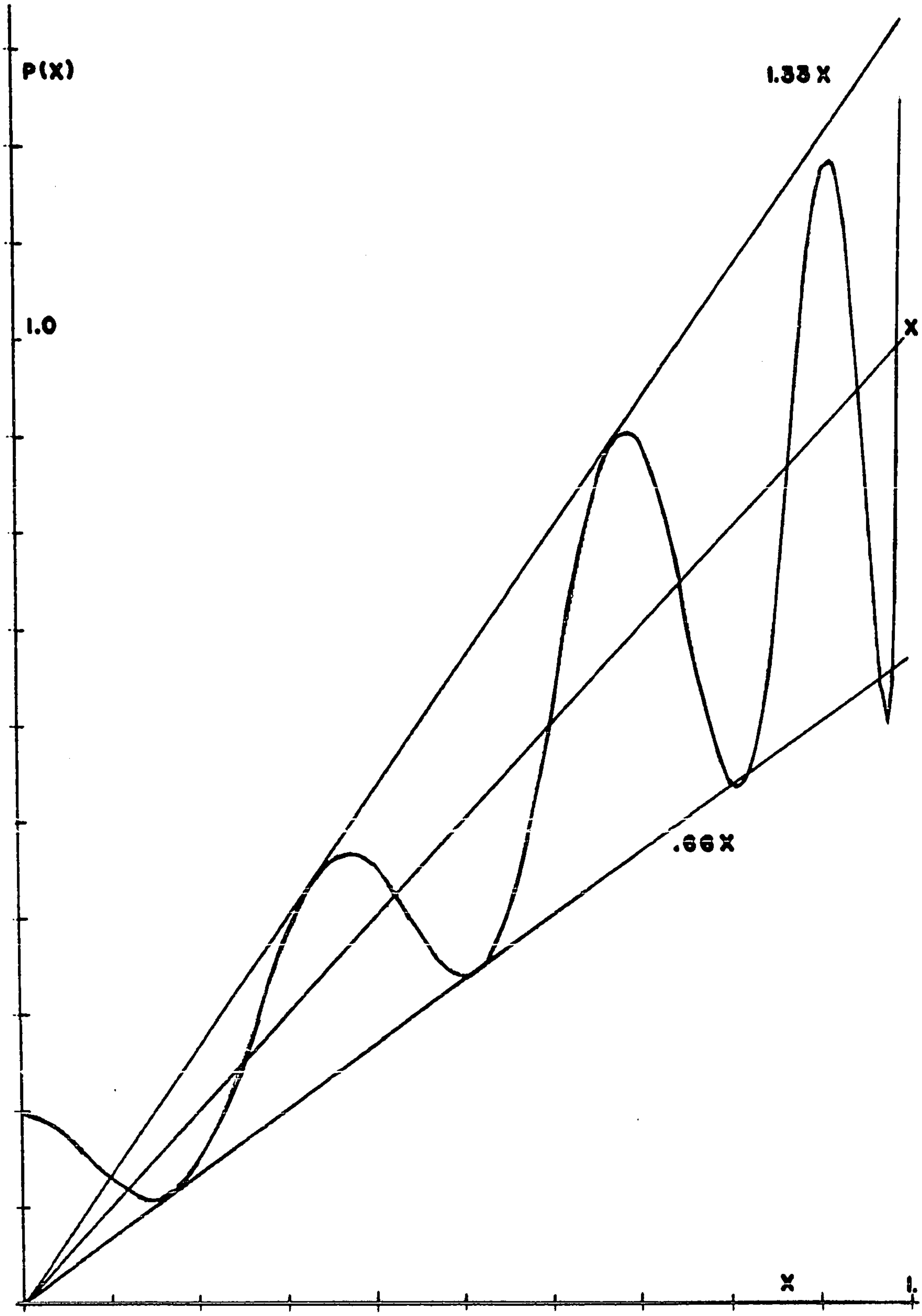


FIGURE 17. APPROXIMATION TO X USING EIGHTEENTH ORDER POLYNOMIAL

and VI with the exception of the special example of this section used to illustrate the extremum approach.

To illustrate the effects of an extremum approach, a family of fourteenth order polynomial approximations to x is plotted in Figure 18. This is a one parameter family of polynomials using the constant term as the arbitrary parameter. To accentuate the effect of the extremum approach the ripple factor was chosen to be 0.5. Typical of the polynomials plotted are

$$\begin{aligned} P_a(x) &= 1453x^{14} - 4460x^{12} + 5129x^{10} - 2697x^{8} \\ &\quad + 617.2x^{6} - 41.88x^{4} + 0.3591x^{2} + 0.15 \\ P_b(x) &= 544.1x^{14} - 1281x^{12} + 765.1x^{10} + 215.7x^{8} \\ &\quad - 344.1x^{6} + 97.74x^{4} - 5.943x^{2} + 0.20 \end{aligned}$$

The values used for all polynomials of this family are $BW = 0.1$ and $R = 0.5$.

There is little noticeable overshooting of the lower limit due to the fact that the polynomial approaches this limit with a slope greater than that of the limit curve. At the upper limit, the slope of the polynomial near the extremum is less than that of the limit curve and noticeable overshooting is present. As the value of x increases, the polynomial becomes more peaked at the approach points. For this reason the overshooting of the upper limit is

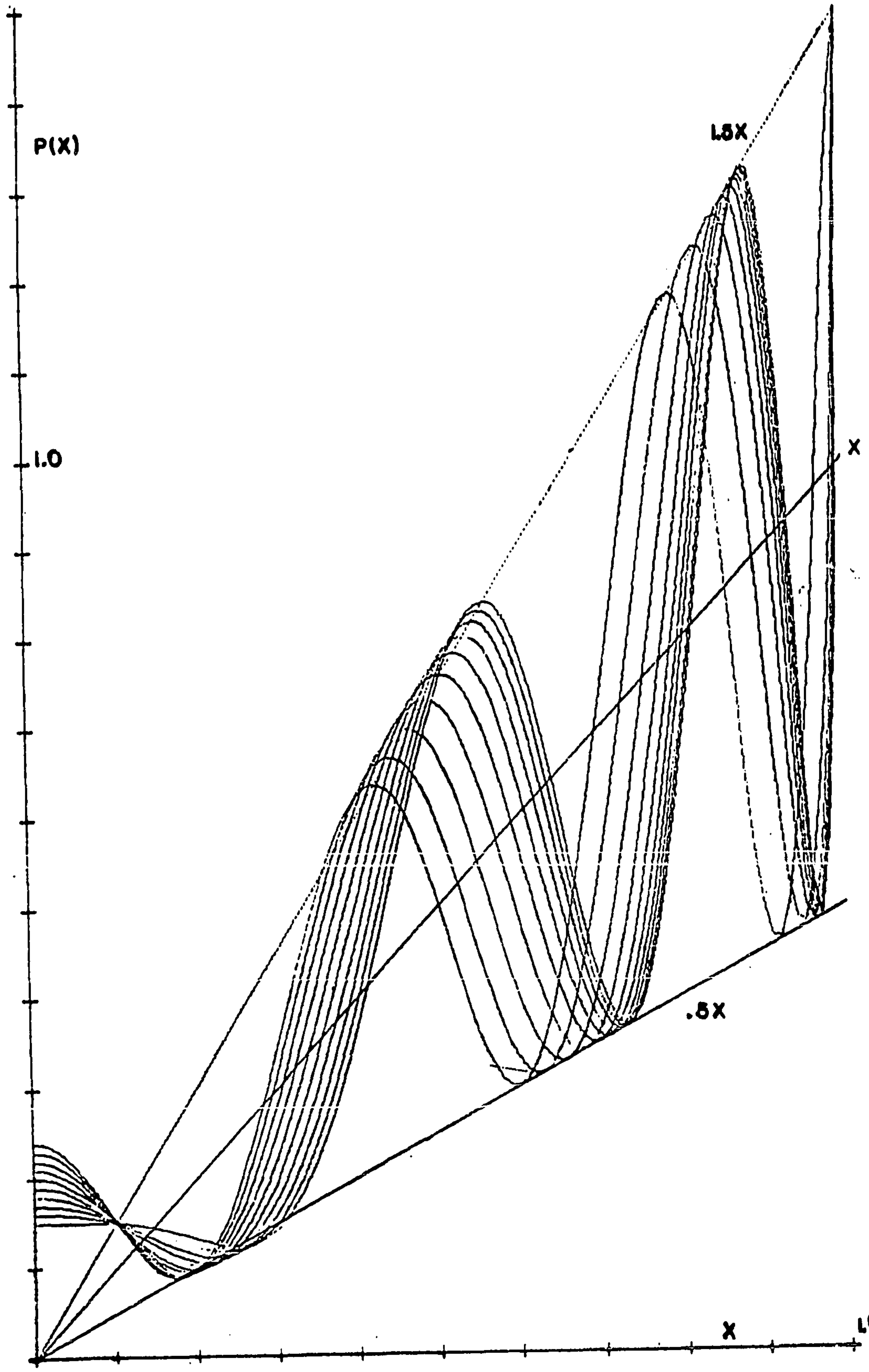


FIGURE 18. APPROXIMATION TO X USING EXTREMUM APPROACH

less apparent for large x, and more apparent overshooting of the lower limit occurs at low values of x. It should be noted that the extremum approach condition is satisfied since the polynomials are equal to the limiting curves at the extrema of the polynomial.

5. Limit definitions

There are two methods of defining the limiting functions to be used in forming an approximating polynomial within the coefficient method. The first of these requires a central function, $f(x)$, and establishes limits as positive and negative deviations from this function. The second method does not require this central function but defines the limiting functions directly. The following examples illustrate the two methods of limit definitions.

The previous examples have all used central functions which were simple polynomials in x and defined the limits about this value. Figures 19 and 20 illustrate approximations to central functions which are not rational functions of x. The function used in Figure 19 is $f(x) = x^{1/2}$ and that used in Figure 20 is $f(x) = x^{1.5}$. Since these are irrational functions of x the limits become irrational functions of x and piece-wise linear techniques were used in the approximation.

Figures 21, 22, and 23 show plots of polynomials formed using the second definition of the limiting

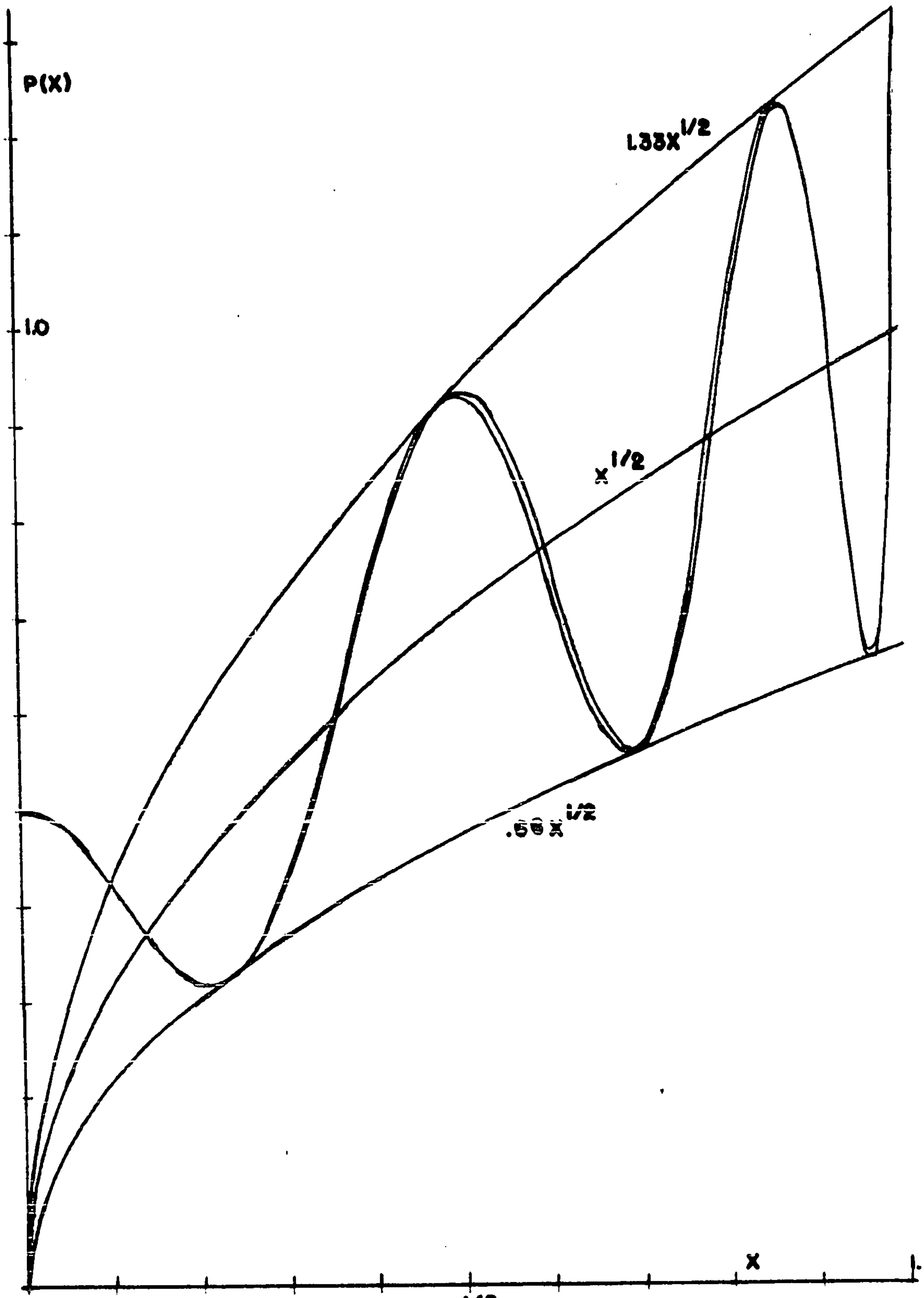


FIGURE 19. APPROXIMATIONS TO $X^{1/2}$

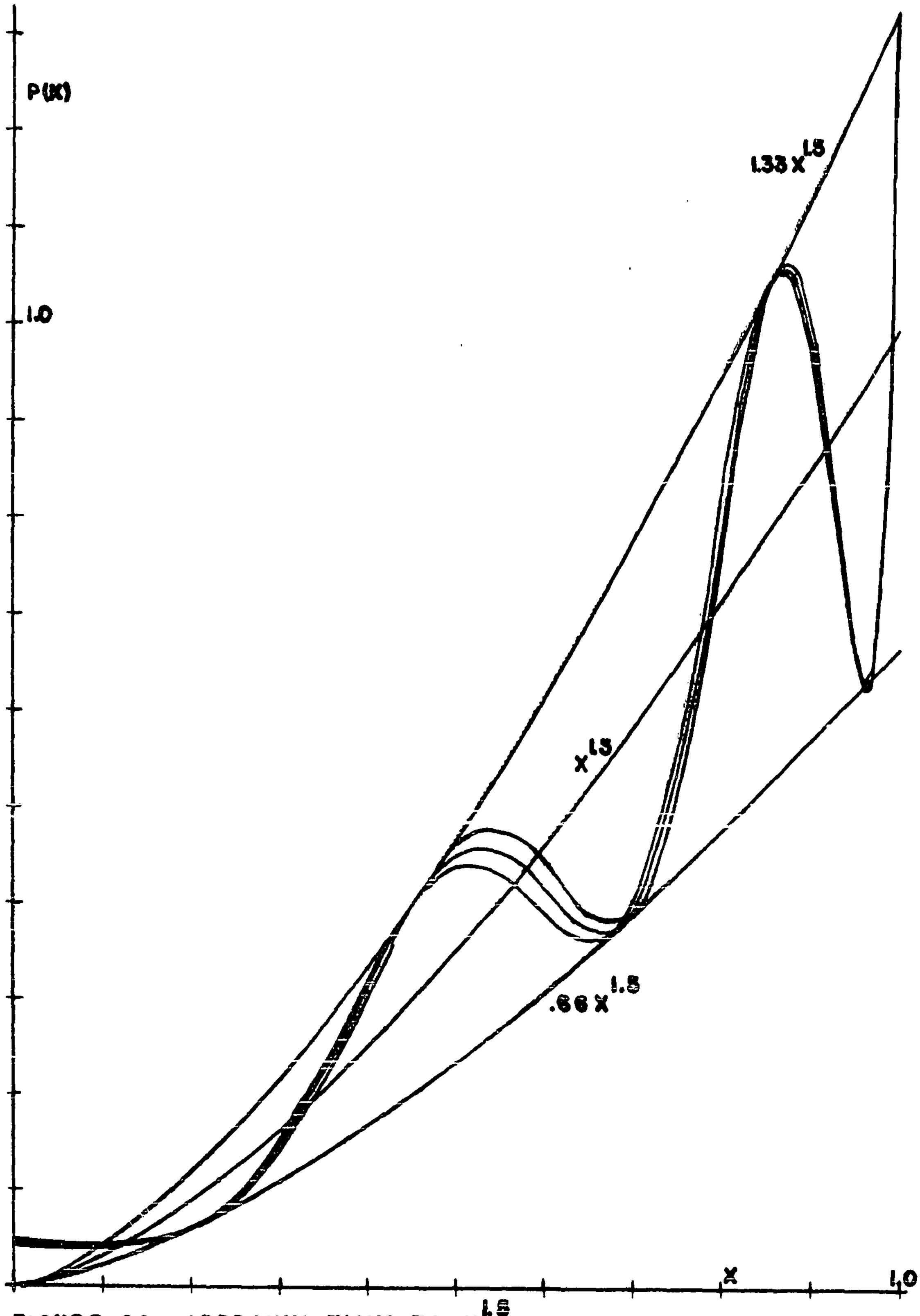


FIGURE 20. APPROXIMATIONS TO $X^{1.5}$

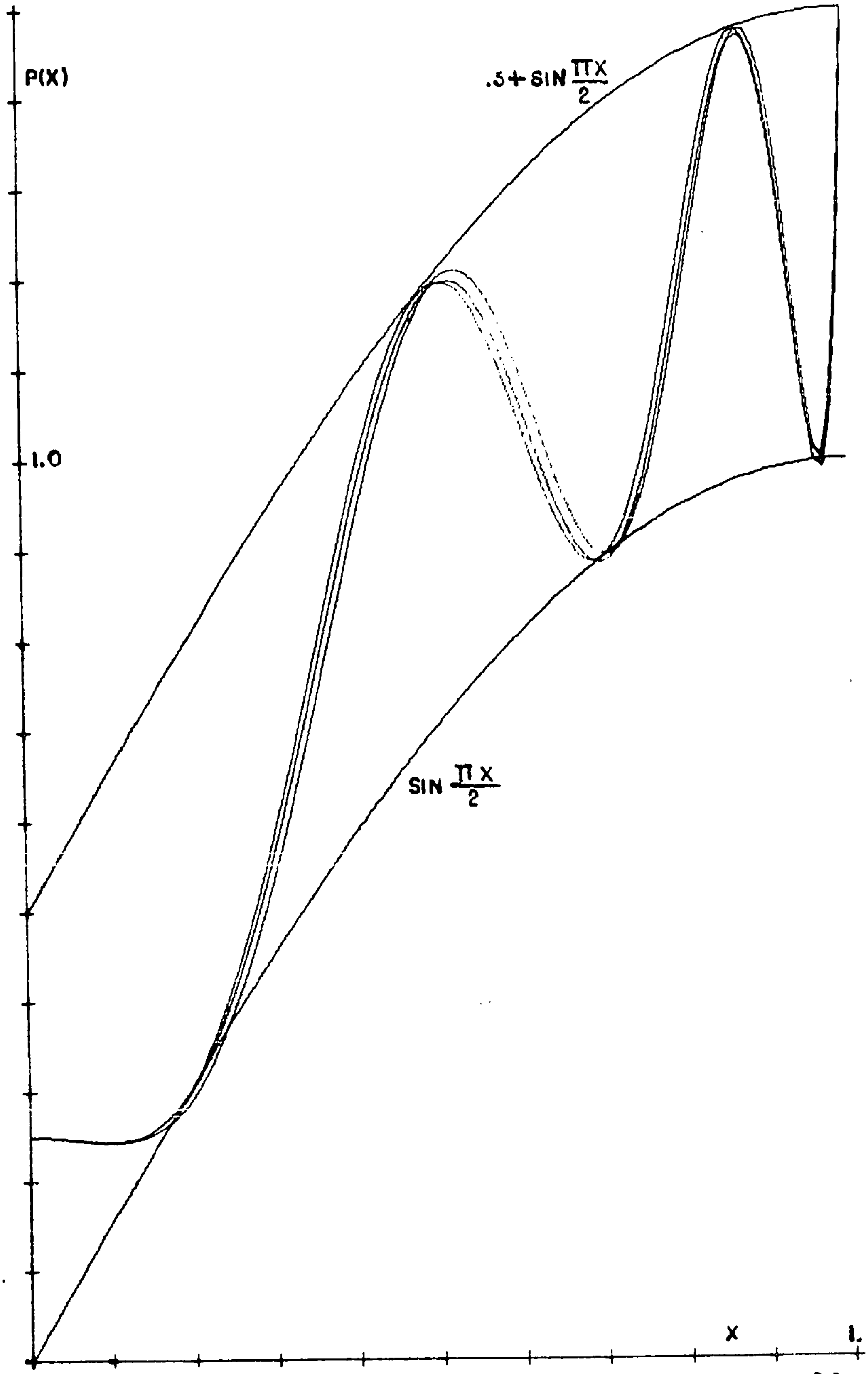


FIGURE 21. POLYNOMIAL FITTING BETWEEN SINE WAVE SEGMENTS

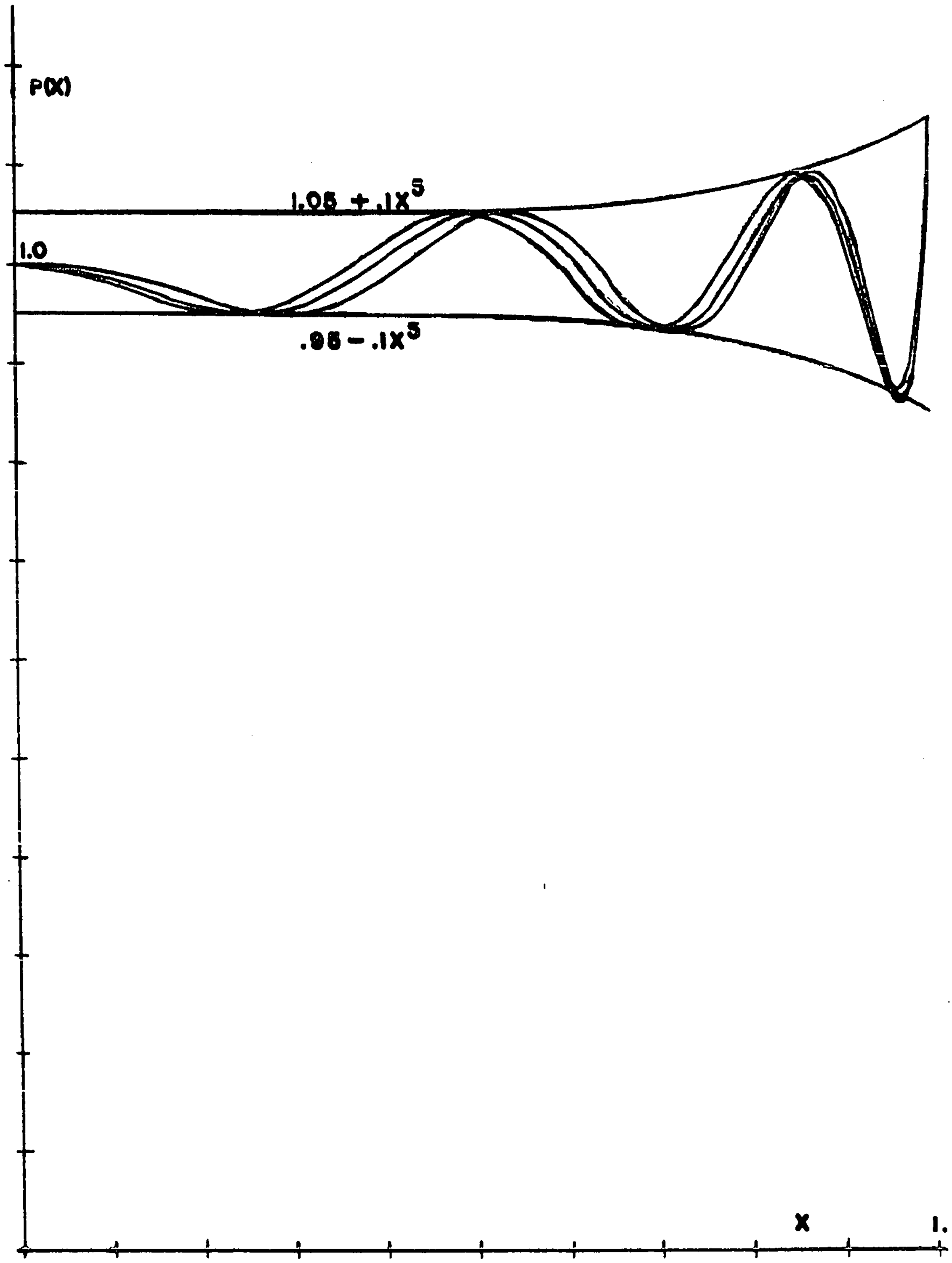


FIGURE 22. POLYNOMIAL FITTING BETWEEN EXPONENTIAL FUNCTIONS

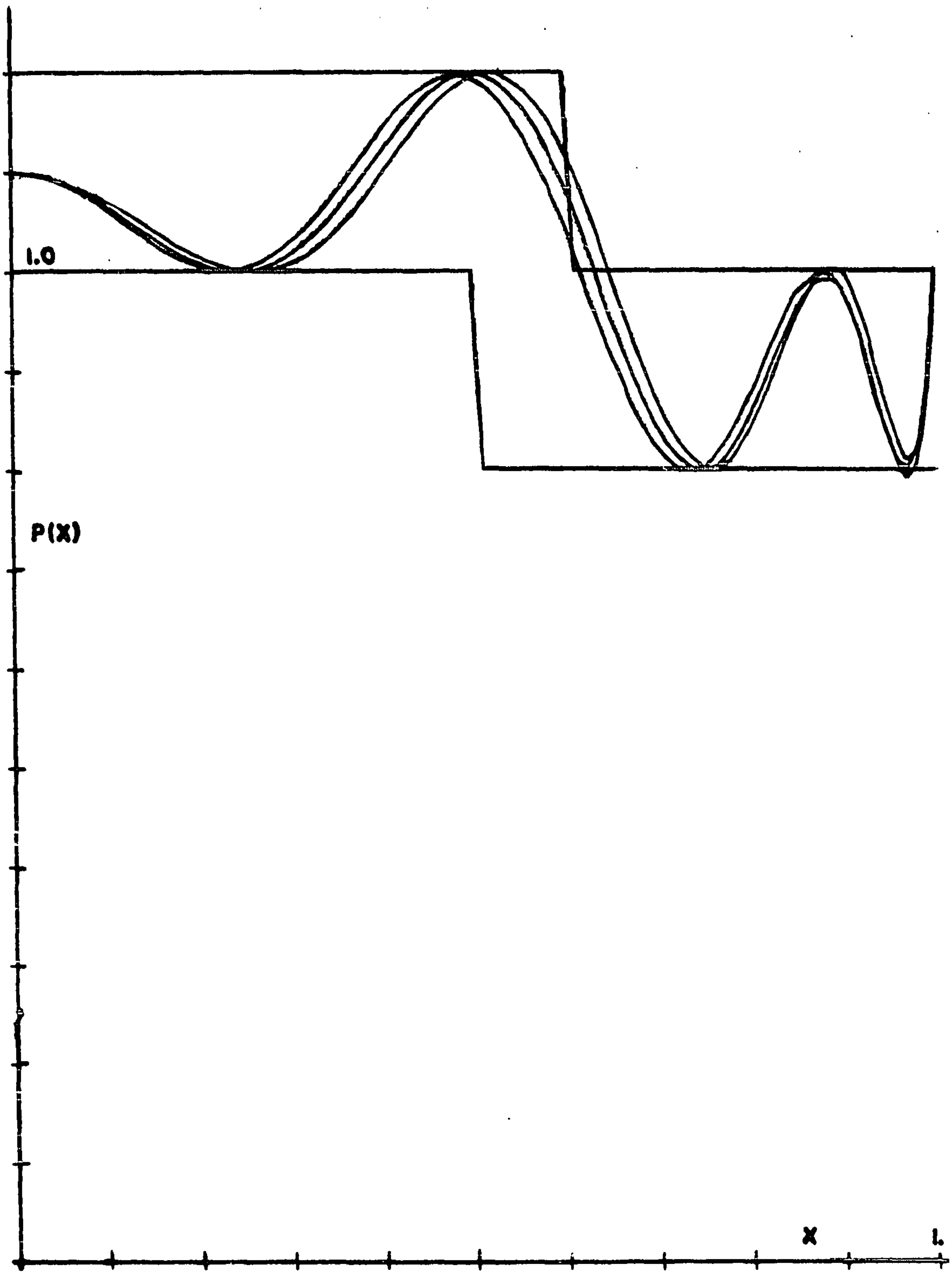


FIGURE 25. POLYNOMIAL FITTING BETWEEN LIMITS HAVING
ABRUPT SLOPE CHANGES

functions. In Figure 21, the limits are defined as sine wave segments. The limits used for forming the curves in Figure 22 are;

$$H_1(x) = 1.05 + 0.1x^5$$

$$H_2(x) = 0.95 - 0.1x^5$$

One weakness of the coefficient method as it is programmed is illustrated by the polynomials plotted in Figure 23. In this example there is an abrupt change in the slope of the limiting curves. In the vicinity of this change in slope, the curves of the polynomials fall outside the limits. Since there is no test to detect this condition in the program, the only means of discovering this type of error is by examination of the curves of the polynomials as they are developed. In this example it was noted that the error became more pronounced as the value of the arbitrary parameter increased. Therefore, one possible correction is the reduction of the value of this parameter.

It is felt that the detection of errors by the examination of the curves as they are plotted is a faster and surer method than is possible by introducing tests for all possible undesirable conditions into the approximation program.

The polynomials plotted in Figure 21 are;

$$P_a(x) = 100x^{16} + 64.14x^{12} - 719.3x^{10} + 952.9x^8$$

$$- 490.4x^6 + 95.64x^4 - 1.637x^2 + 0.25$$

$$P_b(x) = 200x^{14} - 281.22x^{12} - 260.10x^{10} + 660.5x^8$$

$$- 401.3x^6 + 84.69x^4 - 1.390x^2 + 0.25$$

$$P_c(x) = 300x^{14} - 625.2x^{12} + 193.7x^{10} + 377.1x^8$$

$$- 319.0x^6 + 76.14x^4 - 1.415x^2 + 0.25$$

The polynomials plotted in Figure 22 are;

$$P_a(x) = 100x^{14} - 193.9x^{12} + 34.85x^{10} + 145.2x^8$$

$$- 112.5x^6 + 28.67x^4 - 2.169x^2 + 1.00$$

$$P_b(x) = 200x^{14} - 537.6x^{12} + 494.4x^{10} - 150.7x^8$$

$$- 19.34x^6 + 16.08x^4 - 1.701x^2 + 1.00$$

$$P_c(x) = 300x^{14} - 892.2x^{12} + 981.2x^{10} - 477.6x^8$$

$$- 0.0838x^4 + 89.89x^6 - 0.9867x^2 + 1.00$$

The polynomials plotted in Figure 23 are;

$$P_a(x) = 100x^{14} - 137.9x^{12} - 156.6x^{10} + 385.9x^8$$

$$- 246.7x^6 + 59.60x^4 - 4.409x^2 - 1.10$$

$$P_b(x) = 200x^{14} - 493.6x^{12} + 333.8x^{10} + 58.12x^8$$
$$- 140.7x^6 + 45.13x^4 - 3.883x^2 + 1.10$$

$$P_c(x) = 300x^{14} - 854.9x^{12} + 841.8x^{10} + 288.0x^8$$
$$- 240.05x^6 + 28.29x^4 - 3.210x^2 + 1.10$$

In each of the examples of this section the lower end point condition was specified to force the polynomial midway between the limiting functions for x equal to zero.

C. General Remarks

From the results of the examples shown in this section, it may be concluded that the coefficient method does perform as predicted in its development. The examples given are not intended as limitations on the method, but were chosen as typical practical problems. Based on experience gained from the examples solved, several remarks can be made about the possible use of this method.

1. Physical realizability

One of the prime requirements of the polynomials developed using the coefficient method is that they result in functions which are physically realizable as electric networks. Use of the Method of Gewertz should ensure the physical realizability of functions which use the polynomials in the denominator of the appropriate network function. To demonstrate this, each of the polynomials developed in this section was used as the denominator of an approximate network function and synthesized as a lossless ladder network terminated in a one ohm resistor. Each function resulted in a physically realizable network with the exception of the one using the third polynomial, $P_c(x)$, plotted in Figure 22. The function of s formed from this polynomial had only five poles with negative real parts instead of the necessary seven. This was caused by inaccuracies in the factoring routing due to the small coefficient of the x^4 term of this polynomial.

2. Arbitrary parameter values

In several of the examples it was noted that the precision of the approximation was due, in part, to the value chosen for the arbitrary parameter. Experience has shown that the choice of this value affects the sensitivity of the polynomial both to changes in approach points and therefore to the number of iterations required to reach a solution. Even though the process may reach a solution for some randomly chosen value of the arbitrary parameter, there is some optimum range of this value. Within this range, the polynomial is more rapidly formed and is less sensitive to changes in its properties.

The networks formed using the polynomials are also affected by the choice of the arbitrary parameter value.

If the value used is within this range of best values, the network component values are more uniform. There may be a ten to one range in component values. If a value of the arbitrary parameter outside this preferred range is chosen, the ratio of component values may be as high as forty to one. Therefore, in the case of transducer design, it is often necessary to be sure the arbitrary parameter value is within the preferred range for satisfactory designs.

3. Practical applications

The practical applications of the coefficient method are due to its ability to form a family of networks with similar frequency responses and to form polynomial approximations to a wide variety of functions. Section VI illustrates several applications of this method to problems in network synthesis.

VI. APPLICATION TO NETWORK SYNTHESIS PROBLEMS

There are several areas within the field of network synthesis in which the versatility of the coefficient and extremum methods makes it possible to solve problems for which standard approximation techniques are not satisfactory. These are those problems which require constraints other than frequency response characteristics and those which require functions other than simple rational functions to be approximated.

In this section several examples of practical problems which require specifications other than standard frequency response specifications are illustrated. The examples include the use of the coefficient method to control the cut-off characteristic of a network by controlling the slope of the approximating polynomial at the upper end of the pass band. A second example illustrates the possibility of using the coefficient method to allow the synthesis of an all-pole or lossless ladder network to approximate the response of a lattice network which has a zero in the right half of the complex frequency plane. Other applications are discussed but not illustrated by specific examples.

These examples are not intended to include all possible applications of the methods of this thesis, but are typical of the type of problem which may be solved by their application.

A. Control of Response Characteristics

One type of constraint which cannot be included within the usual methods of network approximation is a definite specification on one particular characteristic independent of the other response characteristics. For example, when utilizing Chebyshev Polynomials, once the order of the polynomial to be used has been specified, the cut-off characteristic has also been determined. In certain types of filters it may be desirable to control directly the slope of the cut-off at high frequencies independent of the polynomial order.

The slope of the cut-off character of the filter response is determined by the slope of the approximating polynomial as it passes through the upper end of the pass band. Using the coefficient method, this slope may be controlled by the variation of a combination of parameters. Among those parameters which may be used to control this slope are the coefficient of the highest power term of the polynomial and the polynomial zero nearest the end of the pass band. It is also possible to use the slope of the polynomial itself as the controlling parameter.

The two polynomials given below were formed using the coefficient method to form an equi-ripple approximation to a constant over the range zero to one with the coefficient of x^{12} and the zero nearest one as the

$$542x^{12} - 1934x^{10} + 2657x^{8} - 1727x^{6} + 514x^{4} - 56.4x^{2} + 1.03$$

$$806x^{12} - 2723x^{10} + 3536x^{8} - 2145x^{6} + 599x^{4} - 62.3x^{2} + 1.02$$

arbitrary parameters used to vary the slope at unity. The first polynomial has a slope at unity equal to 31.2 and the second a slope of 138.8. By proper selection of the parameter values a variety of polynomials was formed having slopes lying between these two limits. For comparison purposes the twelfth order Chebyshev Polynomial has a slope of 144 at unity.

Other response characteristics may be controlled by proper definition of the arbitrary parameters. The curves of Figure 13 illustrate the control of network d.c. behavior possible using the band width factor as the controlling variable. Figure 18 illustrates d.c. behavior controlled by variation of the constant term of the polynomial when used as an arbitrary parameter. Combinations of these and other parameters may be used to control other response characteristics.

B. Replacement Networks

It is possible to use the coefficient method to form simple lossless ladder networks which are approximations to more complex networks over a specified frequency range. In some cases this simple network may be used as a replacement for the original network. There are several possible advantages to be gained by this replacement. In general the ladder will have fewer elements than the original network and will contain only passive elements, whereas the original may contain several active elements. Using a ladder to replace a network such as a lattice permits the use of a common ground connection between input and output.

This replacement network may be developed using the coefficient method to approximate a function which is the inverse of the original transfer function. If the original function $T(s)$ is represented as $N(s)/D(s)$ the replacement network will have a transfer function T_1 which may be represented as $1/P(s)$. In this form $P(s)$ is a polynomial approximation to $D(s)/N(s)$ over a specified range of s. The range of approximation must be limited to exclude zeros of the original transfer function.

The data necessary for forming the polynomial approximation may be obtained in one of two ways. The first of these involves using the laboratory or analysis data of the frequency response of the original network. These data must then be squared and inverted to be in the proper form for the approximation technique. The alternate procedure involves the use of the transfer function of the original network. In this process $T(s)$ is multiplied by $T(-s)$ and the product evaluated over the range of s needed for the approximation.

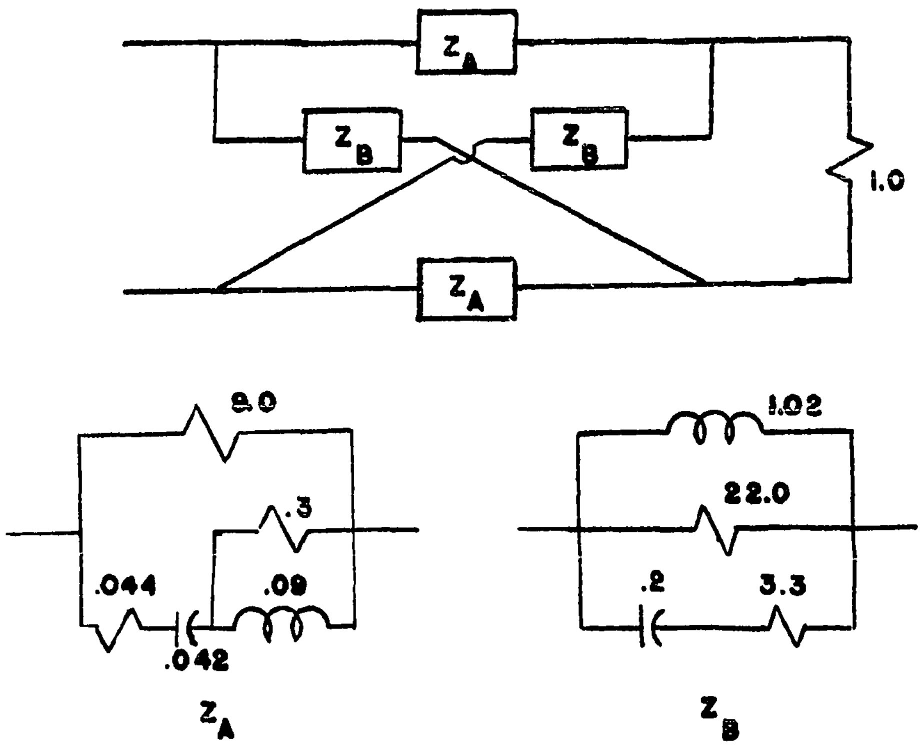
Figure 24a is the schematic diagram of a lattice network which has the transfer function given below. The coefficient

$$\frac{E_2}{E_1} = \frac{.5\ (s^2 + 2s - 8)}{s^2 + 4s + 5}$$

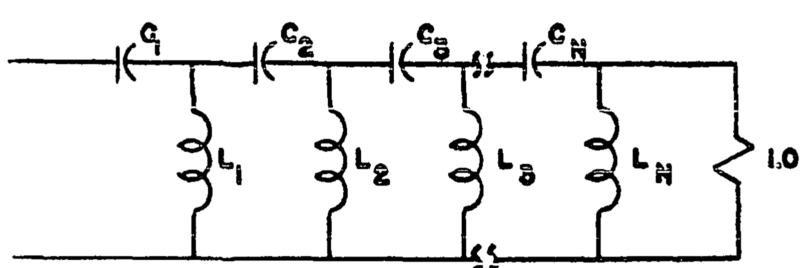
method was used to develop a polynomial approximation over the range of frequency from 0.1 to 1.0 and from that polynomial the ladder networks having the form of Figure 24b were formed. Figure 25 shows the frequency responses of the three networks. The solid line of Figure 25 is the response of the lattice network as determined using digital computer analysis. The circles represent the response data of the network formed using the computer determined response data to develop the approximation polynomial. The crosses represent the response data of a ladder formed using the transfer function of the lattice in the approximation process. The ladder component values for the two cases are given in Table 5. The component values given in this table and those for the lattice network have been scaled for the frequency range zero to one and a one ohm load resistor.

C. Characteristic Matching

Many transducers and other signal sources have frequency-response characteristics which are dependent



A. ORIGINAL LATTICE NETWORK



B. REPLACEMENT LADDER NETWORK

FIGURE 24. EXAMPLE OF NETWORK REPLACEMENT

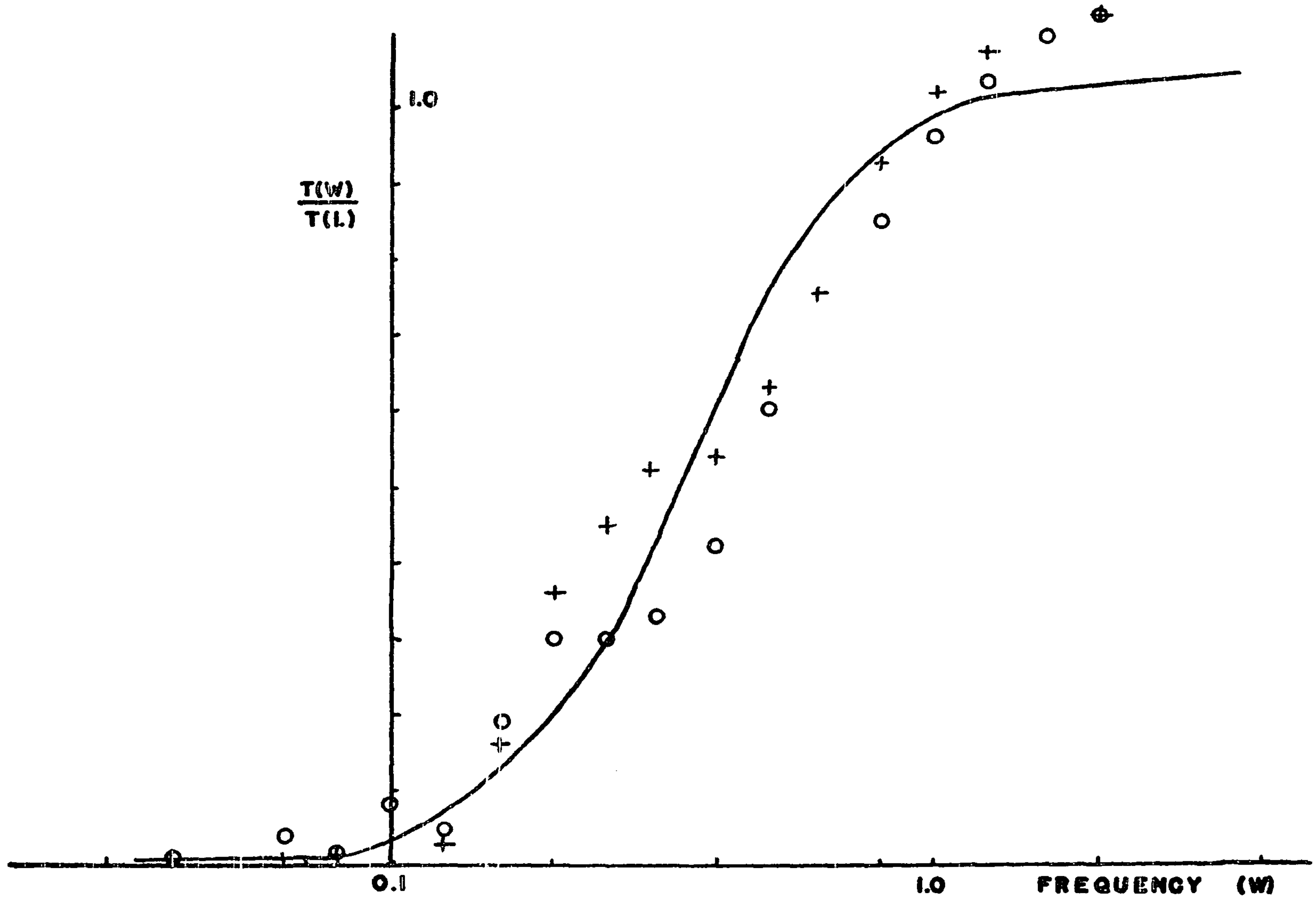


FIGURE 25. RESPONSE OF LATTICE AND REPLACEMENT NETWORKS

Table 5. Ladder component values

Component	Response-derived ladder	Function-derived ladder
L_1	920.0	213.0
C_1	0.0056	0.021
L_2	46.0	22.7
C_2	0.108	0.184
L_3	4.85	4.01
C_3	0.344	0.539
L_4	3.46	1.50
C_4	0.642	0.717
L_5	1.42	1.03
C_5	0.680	0.405

upon the nature of the device and may not be changed. The system in which these devices are used may require other overall frequency characteristics. To accomplish this a filter network must be designed to match the actual device characteristic to the desired response of the system. Therefore the filter network must have a response characteristic which is some combination of these two characteristics. This combination may be a complex function of frequency which is not easily expressed in rational form. Using the coefficient method this combined response function may be approximated by a polynomial even if the function is not a rational form.

One example of the practical application of the coefficient method to the problem of response characteristic matching is the design of an acoustic transducer similar to that described in Section III. In this case it was necessary that the overall system response be an equi-ripple approximation to a constant even though the device response was a three decibel per octave loss with increasing frequency. This required that the transducer have a response which increased three decibels per octave with increasing frequency. The final response function selected involved a frequency to the fourth power term as the numerator and a polynomial approximation to frequency to the fifth power as the denominator. The coefficient method was used to form an equi-ripple fourteenth order polynomial approximation to x^5 over the range of x from 0.1 to 1.0 to be used as the demoninator. This transducer is still in the design stage.

D. Inter-component Constraints

One of the motivating purposes in the development of the approximation methods of this thesis was to permit inter-component constraints to be considered in problems of network synthesis. This application of the methods is fully illustrated by the design outlined in Section III. This particular design uses the extremum method, but the coefficient method could be used alone.

E. Additional Applications

The examples of practical applications of the approximation methods of this thesis described above are typical of those applications which take advantage of the unique properties of these methods. These methods also may be used in any of the filter design problems which are now solved using conventional techniques.

Although the methods of approximation developed were specifically designed for use in network synthesis projects, they are general in nature and should be applicable to any problem requiring polynomials which approximate a given function.

VII. SUMMARY AND CONCLUSIONS

Two methods of forming m-parameter families of approximating polynomials have been developed and illustrated in this thesis. The first of these, the extremum method, is a special purpose method for forming families of polynomials which approximate a constant in an equi-ripple sense over a finite interval. The second method, the coefficient method, is applicable to a wide range of approximation problems. One version of this method forms polynomial approximations to explicit mathematical functions within some variable limits over a prescribed interval. Another version develops polynomials which fit within a pair of arbitrary limiting functions.

A. Basic Algorithm

Although the methods developed in this thesis are different, they do have several basic steps in common. It is possible to derive a general algorithm from these common steps which may be applied to other approximation techniques. The basic steps needed in all methods of this nature are the formation and solution of a system of equations in a set of polynomial characteristics, the application of a second set of conditions to be satisfied by the polynomial, the formation and testing of a trial polynomial and the determination of an improved set of estimates for use in the next cycle of the process.

1. Equation system

The first step in the algorithm requires the formation and solution of a set of equations. The unknowns in this system of equations are a set of basic polynomial characteristics. The extremum method used the extrema of the polynomial as the unknowns. The polynomial characteristics used in the coefficient method were its coefficients. Another possibility would be a system of equations using the zeros of the polynomial as the unknowns. The equation defining the end conditions and the arbitrary parameters of the polynomial family are included within this system of equations.

The coefficients of this system of equations are derived from a set of initial estimates. In the extremum method this set was estimates of the extrema themselves. In the coefficient method, a set of approach point estimates was used to determine the coefficients of the system of equations. There are many such sets which could be used, but for best results the set chosen should have some significance in subsequent calculations.

A trial polynomial is formed from the results of the solution of this set of equations. This polynomial will be used in the following steps of the algorithm.

2. Additional conditions

Once a trial polynomial has been formed, it must be tested to determine if it satisfies a set of constraints. In general, these constraints, which are in addition to the constraints of the system of equations specified in the first step, are related to the limits within which the polynomial must lie.

The additional constraint imposed by the extremum method was that the magnitude of the polynomial at the extremum must equal the ripple magnitude. The coefficient method required the tangents of the polynomial and the limiting function to be equal at the points for which the values of the two functions are equal.

3. Test for solution

In any iterative process there must be some method of testing the trial solution. Once this test is satisfied the trial solution is defined as a final solution. If the trial polynomial does not satisfy the conditions of the test, the iterative process must continue. In this algorithm, these tests usually will be performed on the trial polynomial.

The test used in the extremum method compared the actual values of the trial polynomial at its extrema with the values of the limiting functions at these points. The value used as a measure of the error involved was the sum of the squares of the differences in these values. In the coefficient method, the test involved finding those points for which the tangent of the polynomial equalled the tangents of the limiting functions. These points were compared with the predicted points and the sum of the squares of their differences was used as a measure of convergence.

In each of the methods of this thesis, the points used for this test were related to the points used as initial estimates in the first step. Although this relationship is not a necessity, it is a commonly used practice and permits a new set of estimates to be evaluated easily.

4. Estimate update

Whenever the test of the trial polynomial indicated that no solution has been reached a new set of estimates must be supplied and the cycle repeated. There are many possible means of developing a new set of estimates. The final choice of the method used will depend to a large extent on the definition of the estimates themselves and on the details of the computations in the earlier steps of the algorithm.

In the extremum method, the extrema are used as estimates even though they are never computed within the algorithm. Therefore, a system of nonlinear equations must be solved to update these estimates. It is this process which causes much of the difficulty in obtaining solutions by the extremum method.

The points of approach are used as estimates in each cycle in the coefficient method. The test procedure involves the estimated points of approach and the actual points of equal tangents. Having this set of equal tangent points available, the improved estimates may be derived by linear interpolation using this set and the previous estimates.

The two methods of updating the estimates mentioned above illustrate the range of methods which can be used. In any particular program the updating method must be determined by the details of the program itself.

The four steps described above are not intended as a complete description of an algorithm to produce polynomials which approximate some function. The purpose of the description is to illustrate the similarity of the two methods developed in this thesis and to show that variations on the methods of this thesis can be developed. The basic steps of such algorithms will be those described here, although the details may be quite different.

B. Limitations on the Methods

Except in the case of the extremum method which was designed as a special purpose method to solve one problem, there appear to be no theoretical limitations on the application of the methods of this thesis. Experience has indicated the major source of difficulty to be the design of the computer programs needed to accomplish the computations. With appropriate safeguards built into these programs, there is no reason to limit the use of the methods. However, this safeguard method soon becomes impractical. It appears much more efficient to detect such errors by examination of the polynomial plots. In this way a wide variety of errors may be detected and corrected better than by an array of limited testing routines.

One major solution to possible troubles appears to be the correct selection of the value for the arbitrary parameter. In all examples tested, the methods have performed satisfactorily once a proper choice of this value has been made. These choices are dictated by the behavior of the polynomial itself and may be determined by evaluation of the curves as the polynomials are plotted.

C. Future Investigations

There are several areas which may prove of interest for future investigation. One of the most promising of these is the development of a noniterative method of accomplishing the approximation process. Another promising area is the application of the methods of this thesis to problems of network optimization.

1. Explicit method

Several attempts were made to derive an explicit expression for the approximation. In each case the computations involved were too complicated to permit direct solution. When the assumptions necessary to permit solution were included, the results were too restrictive to be of general use. Because of the need to produce a network meeting certain specifications (Section III), these attempts were not pursued in more detail, but were abandoned in favor of the numerical methods of this thesis.

The attempts made (Appendix, Sections A, B) could serve as starting points for any future attempts to derive an explicit approximation expression. This explicit form should include the desirable features of the iterative forms of this thesis and, in addition, could have the advantage of reduced computation time which is characteristic of many explicit expressions.

2. Network optimization

The optimization technique described by Ishizaki and Watanabe (9) suggests that the approximation procedures of this thesis can be adapted for use in a similar technique. The major differences between the optimization procedure and the approximation procedure are the form of the final solution and application of the output.

The optimization method referenced above is not as general as a method based on the techniques of this thesis could be. The present method is limited to Chebyshev sense optimization. A method based on this thesis could be used for a wide variety of optimization schemes.

3. Other possible areas

The two suggestions for possible future investigation given above are not the only areas available. They are suggested because they represent extensions of the methods derived. Other possible areas of investigation could produce improvements within the framework of the present methods.

Desirable improvements include reprogramming for improved precision or reduced computation time. Other possible improvements are additional polynomial testing and special purpose polynomial factoring subroutine design.

Other fruitful areas for additional investigation may be found in the field of applications. At present, the method has been applied only to the network design problems, but other applications of equal interest may be found.

D. Concluding Remarks

The two methods of forming polynomial families which approximate some desired function or curve were developed to allow more variety in the networks designed to meet a desired frequency response. With a family of networks from which to choose, the choice of a final network may be based upon considerations other than frequency response. In design projects where constraints other than frequency response characteristics were important, the application of the methods of this thesis have produced physically realizable devices meeting all constraints; whereas, the existing approximation techniques did not.

Another degree of flexibility allowed when applying the methods developed in this thesis is the ability to approximate a wider variety of response functions. Using the coefficient method, the choice of functions to be approximated is not limited to simple rational forms, but any finite, single-valued function or curve may be approximated in a form suitable for use in network synthesis problems.

Thus, the methods of this thesis widen the scope of network synthesis methods by permitting the formation of a variety of networks meeting the same specifications and by providing a wider variety of functions which may be used in response specifications.

VIII. BIBLIOGRAPHY

1. Chen, W. H. Linear Network Design and Synthesis. New York, New York, McGraw-Hill Inc. 1964.

2. Winkler, S. The Approximation Problem of Network Synthesis. IRE Transactions on Circuit Theory, CT-1, No. 3: 5-20. September 1954.

3. Butterworth, S. On the Theory of Filter Amplifiers. Experimental Wireless 7: 536-541. October 1920.

4. Olsen, H. F. Elements of Acoustical Engineering. 2nd ed. New York, New York, D. VanNostrand Co., Inc. 1947.

5. Gewertz, C. M. Network Synthesis. Baltimore, Maryland, The Williams and Wilkins Co. 1933.

6. Timan, A. F. Theory of Approximation of Functions of a Real Variable. New York, New York, The Macmillan Co. 1963.

7. Jones, N. B. Lowpass Filters with Approximately Equal-Ripple Modulus Error. Electronics Letters 3, No. 11: 516-517. November 1967.

8. Humpherys, D. S. Rational Function Approximation of Polynomials to Give an Equiripple Error. IEEE Transactions of Circuit Theory, CT-11, No. 4: 479-486. December 1964.

9. Ishizaki, Y. and Watanabe, H. An Iterative Chebyshev Approximation for Network Design. IEEE Transactions on Circuit Theory, CT-15, No. 4: 326-336. December 1968.

10. Ralston, A. A First Course in Numerical Analysis. New York, New York, McGraw-Hill Book Co., Inc. 1955.

11. Remes, E. Sur le calcul effectif des polynomes d'approximation de Tschebyscheff. Compte Rendus Acad. Sci., Paris, No. 199: 337-340. 1934.

IX. APPENDIX

A. Analytic Method

This appendix outlines one unsuccessful attempt to produce an explicit form for the approximation polynomials described in the body of this thesis. Solution of the resultant equations in this development required restrictions which so limited the choice of parameter values as to make the solution impractical.

The analytic method of developing the extremum method is based on the following hypotheses:

Polynomial, $P_n(x)$, is even.

$M = \max|P_n(x)|$ for finite x.

Interval of approximation = $[-1,1]$.

$M = |P_n|$ at n-1 points in the interval and nowhere else along the imaginary axis.

No finite extrema of P_n exist outside the interval of approximation.

From these hypotheses the following root (or zero) locations may be deduced.

Roots of $(M - P_n) = 0$
- end points 2 simple real roots
- $1/2(n-4)$ double real zeros
- 2 simple complex conjugate zeros

Roots of $(M + P_n) = 0$
- $1/2(n-2)$ double real roots
- 2 simple complex conjugate roots

The conjugate simple roots will appear as factors $(x_0^2 + x^2)$

in P_n', $(M - P_n)$, and $(M + P_n)$. Other factors of these terms include:

P_n' includes $(x_1^2 + x^2)$

$(M - P_n)$ includes $(1 - x^2)$ and $(x_2^2 + x^2)$

$(M + P_n)$ includes $(x_3^2 + x^2)$

Since the zeros of P_n' occur at the extrema and the complex roots, the zeros of $(P_n')^2/(x_1^2 + x^2)^2$ are real. Also since $(1 - x^2)(P_n')^2$ contains the same factors as $(M^2 - P_n^2)$, factoring $(P_n')^2/(x_1^2 + x^2)^2$ shows it to have the same factors as $(M - P_n)(M + P_n)/(1 - x^2)(x_2^2 + x^2)(x_3^2 + x^2)$, and all zeros occur at the extrema of P_n.

If $P_n(x)$ is denoted by y, the following equations can be written. The factor, n^2, is included to keep the coefficient of the highest power of x equal to one.

$$(P_n')^2 = \frac{dy}{dx}^2 = \frac{n^2(M^2 - y^2)(x_1^2 + x^2)^2}{(1 - x^2)(x_2^2 + x^2)(x_3^2 + x^2)}$$

$$\frac{dy}{dx} = n(x_1^2 + x^2)\left[\frac{M^2 - y^2}{(1 - x^2)(x_2^2 + x^2)(x_3^2 + x^2)}\right]^{1/2}$$

$$\frac{dy}{(M^2 - y^2)^{1/2}} = \frac{n(x_1^2 + x^2)dx}{[(1 - x^2)(x_2^2 + x^2)(x_3^2 + x^2)]^{1/2}} \qquad (22)$$

Integrating the left-hand side of Equation 22 gives

$$\int \frac{dy}{(M^2 - y^2)^{1/2}} = - \text{arc cos } (\frac{y}{M}) \tag{23}$$

Defining f_n as shown in Equation 24, substituting f_n into Equation 23 and rearranging terms gives Equation 25.

$$f_n = \int \frac{-(x_1{}^2 + x^2)dx}{[(1 - x^2)(x_2{}^2 + x^2)(x_3{}^2 + x^2)]^{1/2}} \tag{24}$$

$$y = M \cos(nf_n) \tag{25}$$

If z is defined as $(x_1{}^2 + x^2)$, x becomes $(z - x_1{}^2)^{1/2}$ and dx becomes $dz/2(z - x_1{}^2)^{1/2}$. Defining μ, B_1, B_2, and B_3 as shown below and substituting them into the equation for f_n gives Equation 26. The denominator of Equation 26 is multiplied through, thus producing the polynomial in μ of Equation 27.

$$\mu = \frac{z}{1 + x_1{}^2}$$

$$B_1 = \frac{x_1{}^2}{1 + x_1{}^2}$$

$$B_2 = \frac{x_1{}^2 - x_2{}^2}{1 + x_1{}^2}$$

$$B_3 = \frac{x_1{}^2 - x_3{}^2}{1 + x_1{}^2}$$

$$f_n = -\frac{1}{4}\int \frac{d(\mu^2)}{[-(1-\mu)(B_1-\mu)(B_2-\mu)(B_3-\mu)]^{1/2}} \tag{26}$$

$$\begin{aligned}(1-\mu)(B_1-\mu)(B_2-\mu)(B_3-\mu) &= B_1B_2B_3 \\ &- \mu(B_1B_2B_3 + B_1B_2 + B_2B_3 + B_3B_1 \\ &+ \mu^2(B_1 + B_2 + B_3 + B_1B_2 + B_2B_3 + B_3B_1) \\ &- \mu^3(1 + B_1 + B_2 + B_3) + \mu^4\end{aligned} \tag{27}$$

One method of handling the integration involved in this definition of f_n is to form an elementary integral. To put Equation 26 into such a form, the coefficients of μ and μ^3 in Equation 27 must be set equal to zero.

$$B_1B_2B_3 + B_1B_2 + B_2B_3 + B_3B_1 = 0$$

$$B_1 + B_2 + B_3 = -1$$

Since this is a set of two equations in three unknowns, no unique solution can be found. One possible trial solution uses large values of x_1 which makes $B_1 = 0$, $B_2 = -1$, and $B_3 = -1$. With these values of the B's substituted into Equation 26, the reduced expression for f_n becomes that shown in Equation 28 which has the indeterminant solution given. Therefore, this assumed solution must be considered invalid.

$$f_n = -\frac{1}{4}\int \frac{d(\mu^2)}{[-(1 - 2\mu^2 + \mu^4)]^{1/2}} \tag{28}$$

$$= \frac{1}{4} \text{ arc cos } [\frac{2(1 - \mu^2)}{0}]$$

As another possible solution, the coefficients for μ and μ^3 are rearranged in the form shown below and set equal to zero.

$$B_1(B_2 + B_3) + B_2B_3(B_1 + 1) = 0$$

$$(B_1 + 1) + (B_2 + B_3) = 0$$

Thus

$$B_2 + B_3 = -(B_1 + 1)$$

and

$$-B_1(B_1 + 1) + B_2B_3(B_1 + 1) = 0$$

Since, by definition, $B_1 = x_1{}^2/(1 + x_1{}^2)$, which is always positive,

$$B_1 = B_2B_3$$

Once again, there are more unknowns than equations so an assumed solution of either $B_2 = -1$ and $B_3 = -B_1$ or $B_2 = -B_1$ and $B_3 = -1$ is substituted into Equation 26 and the equation solved.

$$f_n = -\frac{1}{4}\int \frac{d(\mu^2)}{[-(\mu^4 + (-1 - B_1^2)\mu^2 + B_1^2)]^{1/2}}$$

$$= \frac{1}{4} \text{ arc sin } [\frac{B_1^2 + 1 - 2\mu^2}{B_1^2 - 1}] + cn$$

$$= \frac{1}{4} \text{ arc cos } [\frac{B_1^2 + 1 - 2\mu^2}{B_1^2 - 1}]$$

Substituting this solution into the equation for y, produces the transcendental equation, Equation 29. This equation may be transformed into a polynomial by application of the trigonometric identity, $\cos(n(\text{arc cos } x)) = 1/2\ [(x + (x^2 - 1)^{1/2})^n + (x - (x^2 - 1)^{1/2})^n]$. The polynomial thus produced is a polynomial in u^2 having B_1 as a parameter. This polynomial is shown as Equation 30.

$$y = M \cos[- \quad \text{arc cos}(\frac{B_1^2 + 1 - 2\mu^2}{B_1^2 - 1})] \tag{29}$$

$$y = \frac{M}{2} [\frac{B_1^2 + 1 - 2u^2 + 2(\mu^1 - \mu^2(B_1^2 + 1) + B_1^2)^{1/2}}{B_1^2 - 1}]^{n/4}$$

$$+ \frac{M}{2} [\frac{B_1^2 + 1 - 2\mu^2 - 2(\mu^4 - u^2(B_1^2 + 1) + B_1^2)^{1/2}}{B_1^2 - 1}]^{n/4} \tag{30}$$

When the original definitions for μ and B_1 are substituted into this polynomial, Equation 31 is the result.

For y to be an even polynomial, the coefficients for the

$$y = \frac{M}{2}\left[\frac{T + U^{1/2}}{1 + B_1}\right]^{n/4} + \left[\frac{T - U^{1/2}}{1 + B_1}\right]^{n/4} \tag{31}$$

where

$$T = 2(1 - B_1)x^4 + 4B_1x^2 - (1 + B_1)$$

$$U = 4(1 - B_1)^2x^8 + 16B_1(1 - B_1)x^6 + 4(5B_1^{\ 2} - 1)x^4$$

$$- \ 8B_1(1 + B_1)x^2$$

x^2 and x^6 terms in U must be equal to zero. There are three possible combinations which make the coefficient of x^2 equal to zero:

1. $M = 0$ (not allowed),
2. $B_1 = 0$ (possible),
3. $1 + B_1 = \infty$ (trivial solution).

Therefore to satisfy the hypothesis that P_n be an even polynomial, only one value of B_1 is possible. This violates the hypothesis requiring a variable parameter. Other similar assumptions were also tried but with like results. Each assumption forced a single choice of the arbitrary parameter. Other methods of approach may exist, but were not investigated thoroughly.

B. Second Order Equation Method

A second unsuccessful attempt to find an explicit method of forming the type of approximation described in the thesis is described in this appendix. This method was abandoned because necessary approximations severely restricted the form of the final solution.

This method makes use of the same hypotheses as the method of Appendix, Section A, and is identical to it up to and including Equation 24. In the method of this appendix, a second order differential equation in y is formed. Once formed an attempt is made to solve the equation for y.

The second order differential equation is formed as shown below. Equations 24 and 25 are repeated for reference. In order to simplify writing the equations, a new function, g, is defined by Equation

$$y = M \cos n f_n$$

$$f_n = \int \frac{-(x_1^2 + x^2)\,dx}{[-(1 - x^2)(x_2^2 + x^2)(x_3^2 + x^2)]^{1/2}}$$

$$g = \frac{-(x_1^2 + x^2)}{[-(1 - x^2)(x_2^2 + x^2)(x_3^2 + x^2)]^{1/2}} \qquad (32)$$

$$y = M \cos [n \int gdx]$$

$$y' = ngM \sin [n \int gdx]$$

$$\frac{y'}{g} = nM \sin [n \int gdx]$$

$$[\frac{y'}{g}]' = - n^2Mg \cos [n \int gdx]$$

$$y'' - \frac{y'g'}{g} + n^2g^2y = 0 \qquad (33)$$

The second order differential equation for y in terms of g (Equation 33) must now be solved. When the appropriate operations are performed on the function, g, Equation 33 has the form specified in Equation 34 in which the coefficients are polynomials in x as defined below.

$$Ay'' + By' + Cy = 0 \qquad (34)$$

$$A = (1 - X^2)(x_1^2 + x^2)(x_2^2 + x^2)(x_3^2 + x_2)$$

$$\begin{aligned} -B = \; & x[2(1 - x^2)(x_2^2 + x^2)(x_3^2 + x^2) \\ & - (1 - x^2)(x_1^2 + x^2)(x_2^2 + x^2) \\ & - (1 - x^2)(x_1^2 + x^2)(x_3^2 + x^2) \\ & + (x_1^2 + x^2)(x_2^2 + x^2)(x_3^2 + x^2)] \end{aligned}$$

$$C = n^2(x_1^2 + x^2)^3$$

This equation is now in the appropriate form to be solved using the Method of Frobenius (6). For this method the equation must be in the form specified by Equation 35.

$$My = R(x) \frac{d^2y}{dx^2} + \frac{1}{x} P(x) \frac{dy}{dx} + \frac{1}{x^2} Q(x)y = 0 \qquad (35)$$

where

$$R(x) = A = R_0 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4 + \ldots$$

$$R_0 = 1$$

$$R_1 = 0$$

$$R_2 = \frac{1}{x_3^2} + \frac{1}{x_2^2} + \frac{1}{x_1^2} - 1$$

$$R_3 = 0$$

$$R_4 = \frac{1}{x_2^2 x_3^2} + \frac{1}{x_1^2 x_3^2} + \frac{1}{x_1^2 x_2^2} - \frac{1}{x_3^2} - \frac{1}{x_2^2} - \frac{1}{x_3^2}$$

$$R_5 = 0$$

$$R_6 = \frac{1}{x_1^2 x_2^2 x_3^2} - \frac{1}{x_2^2 x_3^2} - \frac{1}{x_1^2 x_3^2} - \frac{1}{x_1^2 x_2^2}$$

$$R_7 = 0$$

$$R_8 = \frac{-1}{x_1^2 x_2^2 x_3^2}$$

$$P(x) = xB = P_0 + P_1x + P_2x^2 + P_3x^3 + P_4x^4 + \ldots$$

$$P_0 = 0$$

$$P_1 = 0$$

$$P_2 = \frac{-2}{x_1^2} + \frac{1}{x_3^2} + \frac{1}{x_2^2} - 1$$

$$P_3 = 0$$

$$P_4 = \frac{-2}{x_1^2x_2^2} - \frac{2}{x_1^2x_3^2} + \frac{2}{x_2^2x_3^2} + \frac{1}{x_1^2} - \frac{2}{x_2^2} - \frac{2}{x_3^2}$$

$$P_5 = 0$$

$$P_6 = -\frac{2}{x_2^2x_3^2}$$

$$P_7 = 0$$

$$P_8 = \frac{-1}{x_1^2x_2^2x_3^2}$$

$$Q(x) = x^2C = Q_0 + Q_1x + Q_2x^2 + Q_3x^3 + Q_4x^4 + \ldots$$

$$Q_0 = 0$$

$$Q_1 = 0$$

$$Q_2 = \frac{n^2x_1^4}{x_2^2x_3^2}$$

$$Q_3 = 0$$

$$Q_4 = \frac{3n^2 x_1^2}{x_2^2 x_3^2}$$

$$Q_5 = 0$$

$$Q_6 = \frac{3n^2}{x_2^2 x_3^2}$$

$$Q_7 = 0$$

$$Q_8 = \frac{1}{x_1^2 x_2^2 x_3^2}$$

After applying the Method of Frobenius (6), the solution for y has the form of Equation 36. When the appropriate substitutions are made the equation becomes that shown as Equation 37.

$$y = x^s \sum_{k=0}^{\infty} A_k x^k \tag{36}$$

$$\begin{aligned} My = {} & (1 + R_1 x + R_2 x + \ldots)(x(x - 1)A_0 x^{s-2} + (s + 1)sA_1 x^{s-1} \\ & + (s + 2)(s + 1)A_2 x^s + \ldots) + (P_0 + P_1 x + P_2 x^2 + \ldots) \\ & (sA_0 x^{s-2} + (s + 1)A_1 x^{s-1} + (s + 2)Ax^s + \ldots) \\ & + (Q_0 + Q_1 x + Q_2 x^2 + \ldots)(A_0 x^{s-2} + A_1 x^{s-1} + A_2 x^s + \ldots) \end{aligned} \tag{37}$$

Since y must be an even function of x, s cannot be equal to one. If s is set equal to zero, the odd subscripted A terms must be zero. This is already the case without need for the selection of x_2 or x_3.

In order to obtain a recursion formula for the A's in the above equation, Equation 36 is substituted into Equation 34 resulting in Equation 38.

$$A(x) \frac{d^2}{dx^2} (\Sigma A_k x^{2k}) + B(x) \frac{d}{dx} (\Sigma A_k x^{2k}) + C(x) \Sigma A_k x^{2k} = 0 \tag{38}$$

$$\frac{dy}{dx} = \Sigma 2kA_k x^{2k-1}$$

$$\frac{d^2y}{dx^2} = \Sigma 2k(2k - 1)A_k x^{2k-2}$$

For the n^{th} order polynomial specified by the hypothesis, the recursion formulae become those shown below. Therefore, for k to be $(1/2n + 3)$, r_3 must be zero, since $A_{n/2}$ cannot be zero due to the even polynomial hypothesis. Thus, for the value of $k = (1/2n + 3)$, the value for r_3 is given below. This value becomes zero for n = - 6 or - 2, neither of which satisfies the hypothesis.

$$A_{n/2+1} = r_0 A_{n/2} + r_1 A_{n/2-1} + r_2 A_{n/2-2} + r_3 A_{n/2-3} = 0$$

$$A_{n/2+2} = r_1 A_{n/2} + r_2 A_{n/2-1} + r_3 A_{n/2-2} = 0$$

$$A_{n/2+3} = r_2A_{n/2} + r_3A_{n/2-1} = 0$$

$$A_{n/2+4} = r_3A_{n/2} = 0$$

$$r_3 = \frac{4(2k - 3)^2 - n^2}{x_1{}^2x_2{}^2x_3{}^2(4k^2 + 6k + 2)}$$

An alternate possibility is to force the denominator term to approach infinity which may be done by either letting n become infinite, which is impractical, or letting one or more of the x's become infinite. If the latter choice is made, $r_3 = 0$ for any n or k. Therefore, this method does not produce a polynomial satisfying the necessary specifications.